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ABSTRACT

The first of three volumes of a mathematics training course for Navy personnel, this document covers a wide range of basic mathematics. The text begins with number systems, signed numbers, fractions, decimals, and percentages and continues into algebra with exponents, polynomials, and linear equations. Early chapters were designed to give insight into the theories behind computational processes. Logarithms, the slide rule, proportion, and functions are also covered. The final chapters discuss complex numbers, quadratic equations, geometric figures, and trigonometry. Related documents are SE 014 116 and SE 014 117. (JM)

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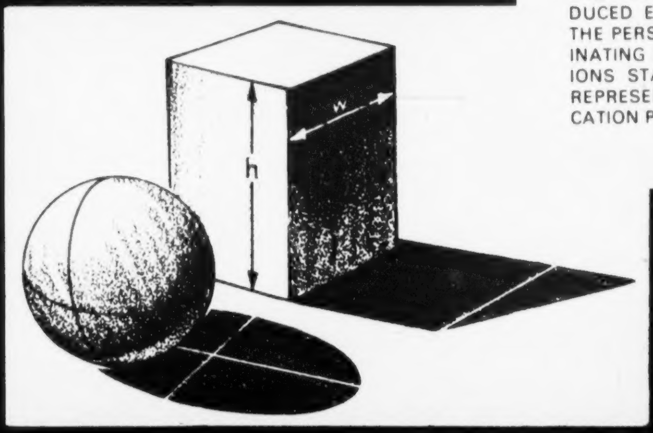

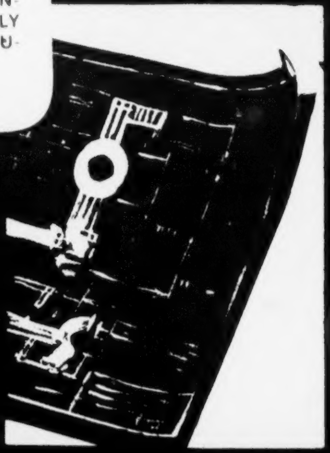

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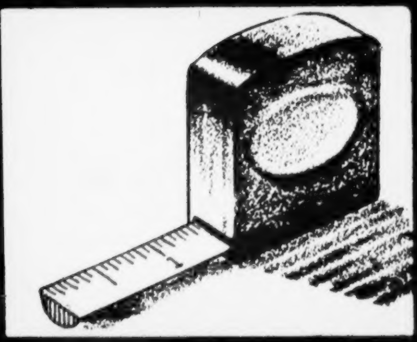

MATHEMATICS, VOL. 1

BUREAU OF NAVAL PERSONNEL
NAVY TRAINING COURSE NAVPERS 10069-C

SE 014 113

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PREFACE

The purpose of this Navy Training Course is to aid those enlisted men who need a basic knowledge of mathematics to carry out their Navy duties. Obviously, to serve the wide variety of ratings needing basic mathematics, the text must be general in nature and is not directed, therefore, toward any one specific rating.

The early chapters that contain basic arithmetic have been designed to give an insight into the theory behind computational processes. Even students who have mastered basic arithmetic rules should find these chapters interesting and useful.

Beginning with chapters on number systems and positive and negative whole numbers, the course continues with discussions of fractions, decimals and percents, exponents, and radicals. Following these topics are chapters concerning common logarithms and the slide rule, algebraic fundamentals, and factoring of polynomials.

Linear equations in one variable and in two variables are discussed in separate chapters, followed by a chapter on ratio, proportion, and variation. Following this are discussions of dependence, functions, and formulas; complex numbers; and quadratic equations. The topics covered in the last three chapters are plane figures, geometric constructions and solid figures, and numerical trigonometry.

This training course was prepared by the United States Navy Training Publications Center, Memphis, Tennessee, for the Bureau of Naval Personnel.

1966 Edition

THE UNITED STATES NAVY

GUARDIAN OF OUR COUNTRY

The United States Navy is responsible for maintaining control of the sea and is a ready force on watch at home and overseas, capable of strong action to preserve the peace or of instant offensive action to win in war.

It is upon the maintenance of this control that our country's glorious future depends; the United States Navy exists to make it so.

WE SERVE WITH HONOR

Tradition, valor, and victory are the Navy's heritage from the past. To these may be added dedication, discipline, and vigilance as the watchwords of the present and the future.

At home or on distant stations we serve with pride, confident in the respect of our country, our shipmates, and our families.

Our responsibilities sober us; our adversities strengthen us.

Service to God and Country is our special privilege. We serve with honor.

THE FUTURE OF THE NAVY

The Navy will always employ new weapons, new techniques, and greater power to protect and defend the United States on the sea, under the sea, and in the air.

Now and in the future, control of the sea gives the United States her greatest advantage for the maintenance of peace and for victory in war.

Mobility, surprise, dispersal, and offensive power are the keynotes of the new Navy. The roots of the Navy lie in a strong belief in the future, in continued dedication to our tasks, and in reflection on our heritage from the past.

Never have our opportunities and our responsibilities been greater.

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* All advancements require commanding officer's recommendation.

† 2 years obligated service required.

INACTIVE DUTY ADVANCEMENT REQUIREMENTS

REQUIREMENTS *		E1 to E2	E2 to E3	E3 to E4	E4 to E5	E5 to E6	E6 to E7	E8	E9
	FOR THESE DRILLS PER YEAR								
TOTAL TIME IN GRADE	48	6 mos.	6 mos.	15 mos.	18 mos.	24 mos.	36 mos.	36 mos.	24 mos.
	24	9 mos.	9 mos.	15 mos.	18 mos.	24 mos.	36 mos.	36 mos.	24 mos.
	NON-DRILLING	12 mos.	24 mos.	24 mos.	36 mos.	48 mos.	48 mos.		
TOTAL TRAINING DUTY IN GRADE †	48	14 days	14 days	14 days	14 days	28 days	42 days	42 days	28 days
	24	14 days	14 days	14 days	14 days	28 days	42 days	42 days	28 days
	NON-DRILLING	None	None	14 days	14 days	28 days	28 days		
PERFORMANCE TESTS		▨	▨	Specified ratings must complete applicable performance tests before taking examination.					
DRILL PARTICIPATION		Satisfactory participation as a member of a drill unit.							
PRACTICAL FACTORS (INCLUDING MILITARY REQUIREMENTS)		Record of Practical Factors, NavPers 760, must be completea for all advancements.							
NAVY TRAINING COURSE (INCLUDING MILITARY REQUIREMENTS)		Completion of applicable course or courses must be entered in service record.							
EXAMINATION		Standard exams are used where available, otherwise locally prepared exams are used.						Standard EXAM, Selection Board.	
AUTHORIZATION		District commandant or CNARESTRA					Bureau of Naval Personnel		

* Recommendation by commanding officer required for all advancements.

† Active duty periods may be substituted for training duty.

CHAPTER 1

NUMBER SYSTEMS AND SETS

Mathematics is a basic tool. Some use of mathematics is found in every rating in the Navy, from the simple arithmetic of counting for inventory purposes to the complicated equations encountered in computer and engineering work. Storekeepers need mathematical computation in their bookkeeping. Damage Controlmen need mathematics to compute stress, centers of gravity, and maximum permissible roll. Electronics principles are frequently stated by means of mathematical formulas. Navigation and engineering also use mathematics to a great extent. As maritime warfare becomes more and more complex, mathematics achieves ever increasing importance as an essential tool.

From the point of view of the individual there are many incentives for learning the subject. Mathematics better equips him to do his present job. It will help him in attaining promotions and the corresponding pay increases. Statistically it has been found that one of the best indicators of a man's potential success as a naval officer is his understanding of mathematics.

This training course begins with the basic facts of arithmetic and continues through some of the early stages of algebra. An attempt is made throughout to give an understanding of why the rules of mathematics are true. This is done because it is felt that rules are easier to learn and remember if the ideas that led to their development are understood.

Many of us have areas in our mathematics background that are hazy, barely understood, or troublesome. Thus, while it may at first seem beneath your dignity to read chapters on fundamental arithmetic, these basic concepts may be just the spots where your difficulties lie. These chapters attempt to treat the subject on an adult level that will be interesting and informative.

COUNTING

Counting is such a basic and natural process that we rarely stop to think about it. The process is based on the idea of ONE-TO-ONE CORRESPONDENCE, which is easily demonstrated by using the fingers. When children count on their fingers, they are placing each finger in one-to-one correspondence with one of the objects being counted. Having outgrown finger counting, we use numerals.

NUMERALS

Numerals are number symbols. One of the simplest numeral systems is the Roman numeral system, in which tally marks are used to represent the objects being counted. Roman numerals appear to be a refinement of the tally method still in use today. By this method, one makes short vertical marks until a total of four is reached; when the fifth tally is counted, a diagonal mark is drawn through the first four marks. Grouping by fives in this way is reminiscent of the Roman numeral system, in which the multiples of five are represented by special symbols.

A number may have many "names." For example, the number 6 may be indicated by any of the following symbols: 9 - 3, 12/2, 5 + 1, or 2 x 3. The important thing to remember is that a number is an idea; various symbols used to indicate a number are merely different ways of expressing the same idea.

POSITIVE WHOLE NUMBERS

The numbers which are used for counting in our number system are sometimes called natural numbers. They are the positive whole numbers, or to use the more precise mathematical term, positive INTEGERS. The Arabic numerals from 0 through 9 are called digits, and an integer may have any number of digits. For example, 5, 32, and 7,049 are all integers. The number of digits in an integer indicates its rank; that is, whether it is "in the hundreds," "in the thousands," etc. The idea of ranking numbers in terms of tens, hundreds, thousands, etc., is based on the PLACE VALUE concept.

PLACE VALUE

Although a system such as the Roman numeral system is adequate for recording the results of counting, it is too cumbersome for purposes of calculation. Before arithmetic could develop as we know it today, the following two important concepts were needed as additions to the counting process:

1. The idea of 0 as a number.
2. Positional notation (place value).

Positional notation is a form of coding in which the value of each digit of a number depends upon its position in relation to the other digits of the number. The convention used in our number system is that each digit has a higher place value than those digits to the right of it.

The place value which corresponds to a given position in a number is determined by the BASE of the number system. The base which is most commonly used is ten, and the system with ten as a base is called the decimal system (decem is the Latin word for ten). Any number is assumed to be a base-ten number, unless some other base is indicated. One exception to this rule occurs when the subject of an entire discussion is some base other than ten. For example, in the discussion of binary (base two) numbers later in this chapter, all numbers are assumed to be binary numbers unless some other base is indicated.

DECIMAL SYSTEM

In the decimal system, each digit position in a number has ten times the value of the position adjacent to it on the right. For example, in the number 11, the 1 on the left is said to be in the "tens place," and its value is 10 times as great as that of the 1 on the right. The 1 on the right is said to be in the "units place," with the understanding that the term "unit" in our system refers to the numeral 1. Thus the number 11 is actually a coded symbol which means "one ten plus one unit." Since ten plus one is eleven, the symbol 11 represents the number eleven.

Figure 1-1 shows the names of several digit positions in the decimal system. If we apply this nomenclature to the digits of the integer 235, then this number symbol means "two hundreds plus three tens plus five units." This number may be expressed in mathematical symbols as follows:

$$2 \times 10 \times 10 + 3 \times 10 \times 1 + 5 \times 1$$

Notice that this bears out our earlier statement: each digit position has 10 times the value of the position adjacent to it on the right.

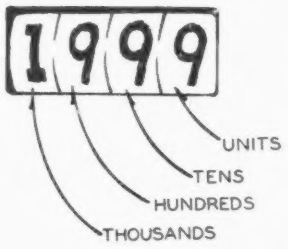


Figure 1-1.—Names of digit positions.

The integer 4,372 is a number symbol whose meaning is "four thousands plus three hundreds plus seven tens plus two units." Expressed in mathematical symbols, this number is as follows:

$$4 \times 1000 + 3 \times 100 + 7 \times 10 + 2 \times 1$$

This presentation may be broken down further, in order to show that each digit position as 10 times the place value of the position on its right, as follows:

$$4 \times 10 \times 100 + 3 \times 10 \times 10 + 7 \times 10 \times 1 + 2 \times 1$$

The comma which appears in a number symbol such as 4,372 is used for "pointing off" the digits into groups of three beginning at the right-hand side. The first group of three digits on the right is the units group; the second group is the thousands group; the third group is the millions group; etc. Some of these groups are shown in table 1-1.

Table 1-1.—Place values and grouping.

Billions group	Millions group	Thousands group	Units group
Hundred billions Ten billions Billions	Hundred millions Ten millions Millions	Hundred thousands Ten thousands Thousands	Hundreds Tens Units

By reference to table 1-1, we can verify that 5,432,786 is read as follows: five million, four hundred thirty-two thousand, seven hundred eighty-six. Notice that the word "and" is not necessary when reading numbers of this kind.

Practice problems:

1. Write the number symbol for seven thousand two hundred eighty-one.
2. Write the meaning, in words, of the symbol 23,469.
3. If a number is in the millions, it must have at least how many digits?
4. If a number has 10 digits, to what number group (thousands, millions, etc.) does it belong?

Answers:

1. 7,281
2. Twenty-three thousand, four hundred sixty-nine.
3. 7
4. Billions

BINARY SYSTEM

The binary number system is constructed in the same manner as the decimal system. However, since the base in this system is two, only two digit symbols are needed for writing numbers. These two digits are 1 and 0. In order to understand why only two digit symbols are needed in the binary system, we may make some observations about the decimal system and then generalize from these.

One of the most striking observations about number systems which utilize the concept of place value is that there is no single-digit symbol for the base. For example, in the decimal system the symbol for ten, the base, is 10. This symbol is compounded from two digit symbols, and its meaning may be interpreted as "one base plus no units." Notice the implication of this where other bases are concerned: Every system uses the same symbol for the base, namely 10. Furthermore, the symbol 10 is not called "ten" except in the decimal system.

Suppose that a number system were constructed with five as a base. Then the only digit symbols needed would be 0, 1, 2, 3, and 4. No single-digit symbol for five is needed, since the symbol 10 in a base-five system with place value means "one five plus no units." In general, in a number system using base N, the largest number for which a single-digit symbol is needed is N minus 1. Therefore, when the base is two the only digit symbols needed are 1 and 0.

An example of a binary number is the symbol 101. We can discover the meaning of this symbol by relating it to the decimal system. Figure 1-2 shows that the place value of each digit position in the binary system is two times the place value of the position adjacent to it on the right. Compare this with figure 1-1, in which the base is ten rather than two.

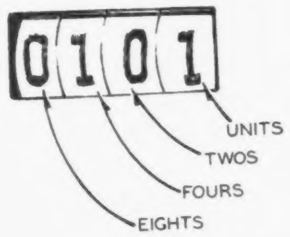


Figure 1-2.—Digit positions in the binary system.

Placing the digits of the number 101 in their respective blocks on figure 1-2, we find that 101 means "one four plus no twos plus one unit." Thus 101 is the binary equivalent of decimal 5. If we wish to convert a decimal number, such as 7, to its binary equivalent, we must break it into parts which are multiples of 2. Since 7 is equal to 4 plus 2 plus 1, we say that it "contains" one 4, one 2, and one unit. Therefore the binary symbol for decimal 7 is 111.

The most common use of the binary number system is in electronic digital computers. All data fed to a typical electronic digital computer is converted to binary form and the computer performs its calculations using binary arithmetic rather than decimal arithmetic. One of the reasons for this is the fact that electrical and electronic equipment utilizes many switching circuits in which there are only two operating conditions. Either the circuit is "on" or it is "off," and a two-digit number system is ideally suited for symbolizing such a situation.

Details concerning binary arithmetic are beyond the scope of this volume, but are available in Mathematics, Volume 3, NavPers 10073, and in Basic Electonics, NavPers 10087-A.

Practice problems:

1. Write the decimal equivalents of the binary numbers 1101, 1010, 1001, and 1111.
2. Write the binary equivalents of the decimal numbers 12, 7, 14, and 3.

Answers:

1. 13, 10, 9, and 15
2. 1100, 111, 1110, and 11

SETS

Any serious study of mathematics leads the student to investigate more than one text and more than one way of approaching each new topic. At the time of printing of this course, much emphasis is being placed on so-called modern math in the public schools. Consequently, the trainee who uses this course is likely to find considerable material, in his parallel reading, which uses the ideas and terminology of the "new" math.

In the following paragraphs, a very brief introduction to some of the set theory of modern math is presented. Although the remainder of this course is not based on set theory, this brief introduction should help in making the transition from traditional methods to newer, experimental methods.

DEFINITIONS AND SYMBOLS

The word "set" implies a collection or grouping of similar objects or symbols. The objects in a set have at least one characteristic in common, such as similarity of appearance or purpose. A set of tools would be an example of a group of objects not necessarily similar in appearance but similar in purpose. The objects or symbols in a set are called members or ELEMENTS of the set.

The elements of a mathematical set are usually symbols, such as numerals, lines, or points. For example, the positive integers greater than zero and less than 5 form a set, as follows:

$$\{1, 2, 3, 4\}$$

Notice that braces are used to indicate sets. This is often done where the elements of the set are not too numerous.

Since the elements of the set $\{2, 4, 6\}$ are the same as the elements of $\{4, 2, 6\}$, these two sets are said to be equal. In other words, equality between sets has nothing to do with the order in which the elements are arranged. Furthermore, repeated elements are not necessary. That is, the elements of $\{2, 2, 3, 4\}$ are simply 2, 3, and 4. Therefore the sets $\{2, 3, 4\}$ and $\{2, 2, 3, 4\}$ are equal.

Practice problems:

1. Use the correct symbols to designate the set of odd positive integers greater than 0 and less than 10.
2. Use the correct symbols to designate the set of names of days of the week which do not contain the letter "s".
3. List the elements of the set of natural numbers greater than 15 and less than 20.
4. Suppose that we have sets as follows:

$A = \{1, 2, 3\}$ $\quad$ $C = \{1, 2, 3, 4\}$

$B = \{1, 2, 2, 3\}$ $\quad$ $D = \{1, 1, 2, 3\}$

Which of these sets are equal?

Answers:

1. $\{1, 3, 5, 7, 9\}$
2. {Monday, Friday}
3. 16, 17, 18, and 19
4. $A = B = D$

SUBSETS

Since it is inconvenient to enumerate all of the elements of a set each time the set is mentioned, sets are often designated by a letter. For example, we could let S represent the set of all positive integers greater than 0 and less than 10. In symbols, this relationship could be stated as follows:

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Now suppose that we have another set, T, which comprises all positive even integers less than 10. This set is then defined as follows:

$$T = \{2, 4, 6, 8\}$$

Notice that every element of T is also an element of S. This establishes the SUBSET relationship; T is said to be a subset of S.

POSITIVE INTEGERS

The most fundamental set of numbers is the set of positive integers. This set comprises the counting numbers (natural numbers) and includes, as subsets, all of the sets of numbers which we have discussed. The set of natural numbers has an outstanding characteristic: it is infinite. This means that the successive elements of the set continue to increase in size without limit, each number being larger by 1 than the number preceding it. Therefore there is no "largest" number; any number that we might choose as larger than all others could be increased to a larger number simply by adding 1 to it.

One way to represent the set of natural numbers symbolically would be as follows:

$$\{1, 2, 3, 4, 5, 6, \ldots\}$$

The three dots, called ellipsis, indicate that the pattern established by the numbers shown continues without limit. In other words, the next number in the set is understood to be 7, the next after that is 8, etc.

POINTS AND LINES

In addition to the many sets which can be formed with number symbols, we frequently find it necessary in mathematics to work with sets composed of points or lines.

A point is an idea, rather than a tangible object, just as a number is. The mark which is made on a piece of paper is merely a symbol representing the point. In strict mathematical terms, a point has no dimensions (physical size) at all. Thus a pencil dot is only a rough picture of a point, useful for indicating the location of the point but certainly not to be confused with the ideal.

Now suppose that a large number of points are placed side by side to form a "string." Picturing this arrangement by drawing dots on paper, we would have a "dotted line." If more dots were placed between the dots already in the string, with the number of dots increasing until we could not see between them, we would have a rough picture of a line. Once again, it is important to emphasize that the picture is only a symbol which represents an ideal line. The ideal line would have length but no width or thickness.

The foregoing discussion leads to the conclusion that a line is actually a set of points. The number of elements in the set is infinite, since the line extends in both directions without limit.

The idea of arranging points together to form a line may be extended to the formation of planes (flat surfaces). A mathematical plane may be considered as the result of placing an infinite number of straight lines side by side, with no space between the lines. Thus the plane is a set of lines. Another way of defining a plane in terms of sets is to consider the plane as the result of placing points side by side in all directions. In this case, the plane is a set of points and the points comprising any line in the plane form a subset.

Line Segments and Rays

When we draw a "line," label its end points A and B, and call it "line AB," we really mean LINE SEGMENT AB. A line segment is a subset of the set of points comprising a line.

When a line is considered to have a starting point but no stopping point (that is, it extends without limit in one direction), it is called a RAY. A ray is not a line segment, because it does not terminate at both ends; it may be appropriate to refer to a ray as a "half-line."

As in the case of a line segment, a ray is a subset of the set of points comprising a line. All three—lines, line segments, and rays—are subsets of the set of points comprising a plane.

THE NUMBER LINE

Among the many devices used for representing a set of numbers, one of the most useful is the number line. To illustrate the construction of a number line, let us place the elements of the set of natural numbers in one-to-one correspondence with points on a line. Since the natural numbers are equally spaced, we select points such that the distances between them are equal. The starting point is labeled 0, the next point is labeled 1, the next 2, etc., using the natural numbers in normal counting order. (See fig. 1-3.) Such an arrangement is often referred to as a scale, a familiar example being the scale on a thermometer.

Thus far in our discussion, we have not mentioned any numbers other than integers. The number line is an ideal device for picturing the

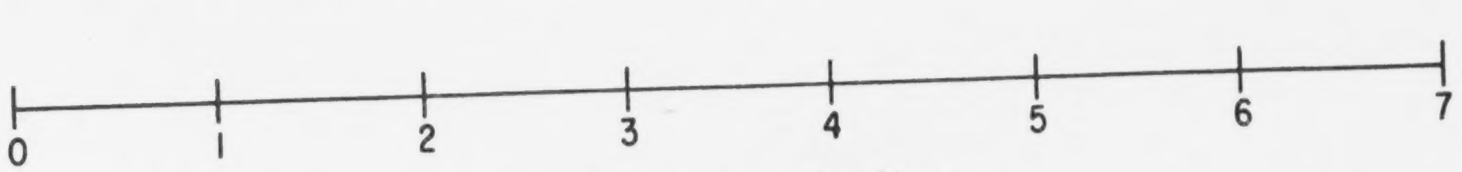


Figure 1-3.—A number line.

relationship between integers and other numbers such as fractions and decimals. It is clear that many points, other than those representing integers, exist on the number line. Examples are the points representing the numbers 1/2 (located halfway between 0 and 1) and 2.5 (located halfway between 2 and 3).

An interesting question arises, concerning the "in-between" points on the number line: How many points (numbers) exist between any two integers? To answer this question, suppose that we first locate the point halfway between 0 and 1, which corresponds to the number 1/2. Then let us locate the point halfway between 0 and 1/2, which corresponds to the number 1/4. The result of the next such halving operation would be 1/8, the next 1/16, etc. If we need more space to continue our halving operations on the number line, we can enlarge our "picture" and then continue.

It soon becomes apparent that the halving process could continue indefinitely; that is, without limit. In other words, the number of points between 0 and 1 is infinite. The same is true of any other interval on the number line. Thus, between any two integers there is an infinite set of numbers other than integers. If this seems physically impossible, considering that even the sharpest pencil point has some width, remember that we are working with ideal points, which have no physical dimensions whatsoever.

Although it is beyond the scope of this course to discuss such topics as orders of infinity, it is interesting to note that the set of integers contains many subsets which are themselves infinite. Not only are the many subsets of numbers other than integers infinite, but also such subsets as the set of all odd integers and the set of all even integers. By intuition we see that these two subsets are infinite, as follows: If we select a particular odd or even integer which we think is the largest possible, a larger one can be formed immediately by merely adding 2.

Perhaps the most practical use for the number line is in explaining the meaning of negative numbers. Negative numbers are discussed in detail in chapter 3 of this course.

CHAPTER 2

POSITIVE INTEGERS

The purpose of this chapter is to review the methods of combining integers. We have already used one combination process in our discussion of counting. We will extend the idea of counting, which is nothing more than simple addition, to develop a systematic method for adding numbers of any size. We will also learn the meaning of subtraction, multiplication, and division.

ADDITION AND SUBTRACTION

In the following discussion, it is assumed that the reader knows the basic addition and subtraction tables, which present such facts as the following: $2 + 3 = 5$, $9 + 8 = 17$, $8 - 3 = 5$, etc.

The operation of addition is indicated by a plus sign (+) as in $8 + 4 = 12$. The numbers 8 and 4 are ADDENDS and the answer (12) is their SUM. The operation of subtraction is indicated by a minus sign (-) as in $9 - 3 = 6$. The number 9 is the MINUEND, 3 is the SUBTRAHEND, and the answer (6) is their DIFFERENCE.

REGROUPING

Addition may be performed with the addends arranged horizontally, if they are small enough and not too numerous. However, the most common method of arranging the addends is to place them in vertical columns. In this arrangement, the units digits of all the addends are alined vertically, as are the tens digits, the hundreds digits, etc. The following example shows three addends arranged properly for addition:

$$\begin{array}{r} 357 \\ 1{,}845 \\ \underline{\quad 22} \end{array}$$

It is customary to draw a line below the last addend, placing the answer below this line. Subtraction problems are arranged in columns in the same manner as for addition, with a line at the bottom and the answer below this line.

Carry and Borrow

Problems involving several addends, with two or more digits each, usually produce sums in one or more of the columns which are greater than 9. For example, suppose that we perform the following addition:

$$\begin{array}{r} 357 \\ 845 \\ 22 \\ \hline 1{,}224 \end{array}$$

The answer was found by a process called "carrying." In this process extra digits, generated when a column sum exceeds 9, are carried to the next column to the left and treated as addends in that column. Carrying may be explained by grouping the original addends. For example, 357 actually means 3 hundreds plus 5 tens plus 7 units. Rewriting the problem with each addend grouped in terms of units, tens, etc., we would have the following:

$$\begin{array}{rcrcr} 300 & + & 50 & + & 7 \\ 800 & + & 40 & + & 5 \\ & & 20 & + & 2 \\ \hline 1{,}100 & + & 110 & + & 14 \end{array}$$

The "extra" digit in the units column of the answer represents 1 ten. We regroup the columns of the answer so that the units column has no digits representing tens, the tens column has no digits representing hundreds, etc., as follows:

$$\begin{aligned} 1{,}100 + 110 + 14 &= 1{,}100 + 110 + 10 + 4 \\ &= 1{,}100 + 120 + 4 \\ &= 1{,}100 + 100 + 20 + 4 \\ &= 1{,}200 + 20 + 4 \\ &= 1{,}000 + 200 + 20 + 4 \\ &= 1{,}224 \end{aligned}$$

When we carry the 10 from the expression $10 + 4$ to the tens column and place it with the 110 to make 120, the result is the same as if we had added 1 to the digits 5, 4, and 2 in the tens column of the original problem. Therefore, the thought process in addition is as follows: Add the 7, 5, and 2 in the units column, getting a sum of 14. Write down the 4 in the units column of the answer and carry the 1 to the tens column. Mentally add the 1 along with the other digits in the tens column, getting a sum of 12. Write down the 2 in the tens column of the answer and carry the 1 to the hundreds column. Mentally add the 1 along with the other digits in the hundreds column, getting a sum of 12. Write down the 2 in the hundreds column of the answer and carry the 1 to the thousands column. If there were other digits in the thousands column to which the 1 could be added, the process would continue as before. Since there are no digits in the thousands column of the original problem, this final 1 is not added to anything, but is simply written in the thousands place in the answer.

The borrow process is the reverse of carrying and is used in subtraction. Borrowing is not necessary in such problems as 46 - 5 and 58 - 53. In the first problem, the thought process may be "5 from 6 is 1 and bring down the 4 to get the difference, 41." In the second problem, the thought process is "3 from 8 is 5" and "5 from 5 is zero," and the answer is 5. More explicitly, the subtraction process in these examples is as follows:

$$\begin{array}{r} 40 + 6 \\ 5 \\ \hline 40 + 1 = 41 \end{array} \qquad \begin{array}{r} 50 + 8 \\ 50 + 3 \\ \hline 0 + 5 = 5 \end{array}$$

This illustrates that we are subtracting units from units and tens from tens.

Now consider the following problem where borrowing is involved:

$$\begin{array}{r} 43 \\ 8 \\ \hline \end{array}$$

If the student uses the borrowing method, he may think "8 from 13 is 5 and bring down 3 to get the difference, 35." In this case what actually was done is as follows:

$$\begin{array}{r} 30 + 13 \\ 8 \\ \hline 30 + 5 = 35 \end{array}$$

A 10 has been borrowed from the tens column and combined with the 3 in the units column to make a number large enough for subtraction of the 8. Notice that borrowing to increase the value of the digit in the units column reduces the value of the digit in the tens column by 1.

Sometimes it is necessary to borrow in more than one column. For example, suppose that we wish to subtract 2,345 from 5,234. Grouping the minuend and subtrahend in units, tens, hundreds, etc., we have the following:

$$\begin{array}{r} 5{,}000 + 200 + 30 + 4 \\ 2{,}000 + 300 + 40 + 5 \\ \hline \end{array}$$

Borrowing a 10 from the 30 in the tens column, we regroup as follows:

$$\begin{array}{r} 5{,}000 + 200 + 20 + 14 \\ 2{,}000 + 300 + 40 + 5 \\ \hline \end{array}$$

The units column is now ready for subtraction. By borrowing from the hundreds column, we can regroup so that subtraction is possible in the tens column, as follows:

$$\begin{array}{r} 5{,}000 + 100 + 120 + 14 \\ 2{,}000 + 300 + 40 + 5 \\ \hline \end{array}$$

In the final regrouping, we borrow from the thousands column to make subtraction possible in the hundreds column, with the following result:

$$\begin{array}{r} 4{,}000 + 1{,}100 + 120 + 14 \\ 2{,}000 + \phantom{1{,}}300 + 40 + 5 \\ \hline 2{,}000 + \phantom{1{,}}800 + 80 + 9 = 2{,}889 \end{array}$$

In actual practice, the borrowing and regrouping are done mentally. The numbers are written in the normal manner, as follows:

$$\begin{array}{r} 5{,}234 \\ -2{,}345 \\ \hline 2{,}889 \end{array}$$

The following thought process is used: Borrow from the tens column, making the 4 become 14. Subtracting in the units column, 5 from 14 is 9. In the tens column, we now have a 2 in the minuend as a result of the first borrowing operation. Some students find it helpful at first to cancel any digits that are reduced as a result of borrowing, jotting down the digit of next lower value just above the canceled digit. This has been done in the following example:

$$\begin{array}{r} 4\ \ 12\ \ \\ \cancel{5},\cancel{2}\cancel{3}\cancel{4} \\ \underline{-2,345} \\ 2,889 \end{array}$$

After canceling the 3, we proceed with the subtraction, one column at a time. We borrow from the hundreds column to change the 2 that we now have in the tens column into 12. Subtracting in the tens column, 4 from 12 is 8. Proceeding in the same way for the hundreds column, 3 from 11 is 8. Finally, in the thousands column, 2 from 4 is 2.

Practice problems. In problems 1 through 4, add the indicated numbers. In problems 5 through 8, subtract the lower number from the upper.

1. Add 23, 468, 7, and 9,045.

$$\begin{array}{llllllll} 2. & 129 & 3. & 9,497 & 4. & 67,856 & & \\ & 958 & & 6,364 & & 22,851 & & \\ & 787 & & 4,269 & & 44,238 & & \\ & \underline{436} & & \underline{9,785} & & \underline{97,156} & & \\ \\ 5. & 709 & 6. & 8,700 & 7. & 7,928 & 8. & 75,168 \\ & \underline{594} & & \underline{5,008} & & \underline{5,349} & & \underline{28,089} \end{array}$$

Answers:

1. 9,543	2. 2,310	3. 29,915	4. 232,101
5. 115	6. 3,692	7. 2,579	8. 47,079

Denominate Numbers

Numbers that have a unit of measure associated with them, such as yard, kilowatt, pound, pint, etc., are called DENOMINATE NUMBERS. The word "denominate" means the numbers have been given a name; they are not just abstract symbols. To add denominate numbers, add all units of the same kind. Simplify the result, if possible. The following example illustrates the addition of 6 ft 8 in. to 4 ft 5 in.:

$$\begin{array}{rr} 6\text{ ft} & 8\text{ in.} \\ \underline{4\text{ ft}} & \underline{5\text{ in.}} \\ 10\text{ ft} & 13\text{ in.} \end{array}$$

Since 13 in. is the equivalent of 1 ft 1 in., we regroup the answer as 11 ft 1 in.

A similar problem would be to add 20 degrees 44 minutes 6 seconds to 13 degrees 22 minutes 5 seconds. This is illustrated as follows:

$$\begin{array}{rrr} 20\text{ deg} & 44\text{ min} & 6\text{ sec} \\ \underline{13\text{ deg}} & \underline{22\text{ min}} & \underline{5\text{ sec}} \\ 33\text{ deg} & 66\text{ min} & 11\text{ sec} \end{array}$$

This answer is regrouped as 34 deg 6 min 11 sec.

Numbers must be expressed in units of the same kind, in order to be combined. For instance, the sum of 6 kilowatts plus 1 watt is not 7 kilowatts nor is it 7 watts. The sum can only be indicated (rather than performing the operation) unless some method is used to write these numbers in units of the same value.

Subtraction of denominate numbers also involves the regrouping idea. If we wish to subtract 16 deg 8 min 2 sec from 28 deg 4 min 3 sec, for example, we would have the following arrangement:

$$\begin{array}{rrr} 28\text{ deg} & 4\text{ min} & 3\text{ sec} \\ \underline{-16\text{ deg}} & \underline{8\text{ min}} & \underline{2\text{ sec}} \end{array}$$

In order to subtract 8 min from 4 min we regroup as follows:

$$\begin{array}{rrr} 27\text{ deg} & 64\text{ min} & 3\text{ sec} \\ \underline{-16\text{ deg}} & \underline{8\text{ min}} & \underline{2\text{ sec}} \\ 11\text{ deg} & 56\text{ min} & 1\text{ sec} \end{array}$$

Practice problems. In problems 1 and 2, add. In problems 3 and 4, subtract the lower number from the upper.

$$\begin{array}{lrrr} 1. & 6\text{ yd} & 2\text{ ft} & 7\text{ in.} \\ & & 1\text{ ft} & 9\text{ in.} \\ & \underline{2\text{ yd}} & \underline{\phantom{2\text{ ft}}} & \underline{10\text{ in.}} \end{array}$$

$$\begin{array}{lrrr} 2. & 9\text{ hr} & 47\text{ min} & 51\text{ sec} \\ & 3\text{ hr} & 36\text{ min} & 23\text{ sec} \\ & \underline{5\text{ hr}} & \underline{15\text{ min}} & \underline{23\text{ sec}} \end{array}$$

$$\begin{array}{lrrr} 3. & 15\text{ hr} & 25\text{ min} & 10\text{ sec} \\ & \underline{-6\text{ hr}} & \underline{50\text{ min}} & \underline{35\text{ sec}} \end{array}$$

$$\begin{array}{lrrr} 4. & 125\text{ deg} & & \\ & \underline{47\text{ deg}} & \underline{9\text{ min}} & \underline{14\text{ sec}} \end{array}$$

Answers:

1. 9 yd 2 ft 2 in.
2. 18 hr 39 min 37 sec
3. 8 hr 34 min 35 sec
4. 77 deg 50 min 46 sec

Mental Calculation

Mental regrouping can be used to avoid the necessity of writing down some of the steps, or of rewriting in columns, when groups of one-digit or two-digit numbers are to be added or subtracted.

One of the most common devices for rapid addition is recognition of groups of digits whose sum is 10. For example, in the following problem two "ten groups" have been marked with braces:

$$\begin{array}{l} 7 \\ \left.\begin{array}{l}6\\4\end{array}\right\} 10 \\ 5 \\ \left.\begin{array}{l}1\\9\end{array}\right\} 10 \\ \hline \end{array}$$

To add this column as grouped, you would say to yourself, "7, 17, 22, 32." The thought should be just the successive totals as shown above and not such cumbersome steps as "7 + 10, 17, + 5, 22, + 10, 32."

When successive digits appear in a column and their sum is less than 10, it is often convenient to think of them, too, as a sum rather than separately. Thus, if adding a column in which the sum of two successive digits is 10 or less, group them as follows:

$$\begin{array}{l} \left.\begin{array}{l}3\\1\\1\end{array}\right\} 5 \\ \left.\begin{array}{l}8\\1\end{array}\right\} 9 \\ \left.\begin{array}{l}4\\6\end{array}\right\} 10 \\ \hline \end{array}$$

The thought process here might be, as shown by the grouping, "5, 14, 24."

Practice problems. Add the following columns from the top down, as in the preceding example:

1. 2	2. 4	3. 88	4. 57
7	6	36	32
3	7	59	64
6	8	82	97
4	1	28	79
1	8	57	44

Answers, showing successive mental steps:

1. 2, 12, 22, 23 - - Final answer, 23
2. 10, 17, 26, 34 - - Final answer, 34
3. Units column: 14, 23, 33, 40 - - Write down 0, carry 4.
   Tens column: 12, 20, 30, 35 - - Final answer, 350.
4. Units column: 9, 20, 29, 33 - - Write down 3, carry 3.
   Tens column: 8, 17, 26, 37 - - Final answer, 373.

SUBTRACTION.—In an example such as 73 - 46, the conventional approach is to place 46 under 73 and subtract units from units and tens from tens, and write only the difference without the intermediate steps. To do this, the best method is to begin at the left. Thus, in the example 73 - 46, we take 40 from 73 and then take 6 from the result. This is done mentally, however, and the thought would be "73, 33, 27," or "33, 27." In the example 84 - 21 the thought is "64, 63" and in the example 64 - 39 the thought is "34, 25."

Practice problems. Mentally subtract and write only the difference:

1. 47 - 24
2. 69 - 38
3. 87 - 58
4. 86 - 73
5. 82 - 41
6. 30 - 12

Answers, showing successive mental steps:

1. 27, 23 - - Final answer, 23
2. 39, 31 - - Final answer, 31
3. 37, 29 - - Final answer, 29
4. 16, 13 - - Final answer, 13
5. 42, 41 - - Final answer, 41
6. 20, 18 - - Final answer, 18

MULTIPLICATION AND DIVISION

Multiplication may be indicated by a multiplication sign (x) between two numbers, a dot between two numbers, or parentheses around one or both of the numbers to be multiplied. The following examples illustrate these methods:

$$6 \times 8 = 48$$
$$6 \cdot 8 = 48$$
$$6(8) = 48$$
$$(6)(8) = 48$$

Notice that when a dot is used to indicate multiplication, it is distinguished from a decimal point or a period by being placed above the line of writing, as in example 2, whereas a period or decimal point appears on the line. Notice also that when parentheses are used to indicate multiplication, the numbers to be multiplied are spaced closer together than they are when the dot or x is used.

In each of the four examples just given, 6 is the MULTIPLIER and 8 is the MULTIPLICAND. Both the 6 and the 8 are FACTORS, and the more modern texts refer to them this way. The "answer" in a multiplication problem is the PRODUCT; in the examples just given, the product is 48.

Division usually is indicated either by a division sign (÷) or by placing one number over another number with a line between the numbers, as in the following examples:

1. $8 \div 4 = 2$
2. $\frac{8}{4} = 2$

The number 8 is the DIVIDEND, 4 is the DIVISOR, and 2 is the QUOTIENT.

MULTIPLICATION METHODS

The multiplication of whole numbers may be thought of as a short process of adding equal numbers. For example, 6(5) and 6 x 5 are read as six 5's. Of course we could write 5 six times and add, but if we learn that the result is 30 we can save time. Although the concept of adding equal numbers is quite adequate in explaining multiplication of whole numbers, it is only a special case of a more general definition, which will be explained later in multiplication involving fractions.

Grouping

Let us examine the process involved in multiplying 6 times 27 to get the product 162. We first arrange the factors in the following manner:

$$\begin{array}{r} 27 \\ \text{x}6 \\ \hline 162 \end{array}$$

The thought process is as follows:

1. 6 times 7 is 42. Write down the 2 and carry the 4.
2. 6 times 2 is 12. Add the 4 that was carried over from step 1 and write the result, 16, beside the 2 that was written in step 1.
3. The final answer is 162.

Table 2-1 shows that the factors were grouped in units, tens, etc. The multiplication was done in three steps: Six times 7 units is 42 units (or 4 tens and 2 units) and six times 2 tens is 12 tens (or 1 hundred and 2 tens). Then the tens were added and the product was written as 162.

Table 2-1.—Multiplying by a one-digit number.

	Hundreds	Tens	Units
6(27) = 162		2	7
			6
		4	2
	1	2	
	1	6	2

In preparing numbers for multiplication as in table 2-1, it is important to place the digits of the factors in the proper columns; that is, units must be placed in the units column, tens in tens column, and hundreds in hundreds column. Notice that it is not necessary to write the zero in the case of 12 tens (120) since the 1 and 2 are written in the proper columns. In practice, the addition is done mentally, and just the product is written without the intervening steps.

Multiplying a number with more than two digits by a one-digit number, as shown in table 2-2, involves no new ideas. Three times 6 units is 18 units (1 ten and 8 units), 3 times 0 tens is 0, and 3 times 4 hundreds is 12 hundreds (1

Table 2-2.—Multiplying a three-digit number by a one-digit number.

	Thousands	Hundreds	Tens	Units
3(406) = 1,218		4	0	6 3
			1	8
	1	2		
	1	2	1	8

thousand and 2 hundreds). Notice that it is not necessary to write the 0's resulting from the step "3 times 0 tens is 0." The two terminal 0's of the number 1,200 are also omitted, since the 1 and the 2 are placed in their correct columns by the position of the 4.

Partial Products

In the example, 6(8) = 48, notice that the multiplying could be done another way to get the correct product as follows:

$$6(3 + 5) = 6 \times 3 + 6 \times 5$$

That is, we can break 8 into 3 and 5, multiply each of these by the other factor, and add the partial products. This idea is employed in multiplying by a two-digit number. Consider the following example:

$$\begin{array}{r} 43 \\ \underline{\times 27} \\ 1,161 \end{array}$$

Breaking the 27 into 20 + 7, we have 7 units times 43 plus 2 tens times 43, as follows:

$$43(20 + 7) = (43)(7) + (43)(20)$$

Since 7 units times 43 is 301 units, and 2 tens times 43 is 86 tens, we have the following:

$$\begin{array}{rl} 43 & \\ \underline{\times 27} & \\ 301 & = \text{3 hundreds, 0 tens, 1 unit} \\ \underline{86} & = \text{8 hundreds, 6 tens} \\ 1,161 & \end{array}$$

As long as the partial products are written in the correct columns, we can multiply beginning from either the left or the right of the multiplier. Thus, multiplying from the left, we have

$$\begin{array}{r} 43 \\ \underline{\times 27} \\ 86 \\ \underline{301} \\ 1,161 \end{array}$$

Multiplication by a number having more places involves no new ideas.

End Zeros

The placement of partial products must be kept in mind when multiplying in problems involving end zeros, as in the following example:

$$\begin{array}{r} 27 \\ \underline{\times 40} \\ 1,080 \end{array}$$

We have 0 units times 27 plus 4 tens times 27, as follows:

$$\begin{array}{r} 27 \\ \underline{\times 40} \\ 0 \\ \underline{108} \\ 1,080 \end{array}$$

The zero in the units place plays an important part in the reading of the final product. End zeros are often called "place holders" since their only function in the problem is to hold the digit positions which they occupy, thus helping to place the other digits in the problem correctly.

The end zero in the foregoing problem can be accounted for very nicely, while at the same time placing the other digits correctly, by means of a shortcut. This consists of offsetting the 40 one place to the right and then simply bringing down the 0, without using it as a multiplier at all. The problem would appear as follows:

$$\begin{array}{r} 27 \\ \underline{\times 40} \\ 1,080 \end{array}$$

If the problem involves a multiplier with more than one end 0, the multiplier is offset as many places to the right as there are end 0's. For example, consider the following multiplication in which the multiplier, 300, has two end 0's:

$$\begin{array}{r} 220 \\ \underline{\times 300} \\ 66,000 \end{array}$$

Notice that there are as many place-holding zeros at the end in the product as there are place-holding zeros in the multiplier and the multiplicand combined.

Placement of Decimal Points

In any whole number in the decimal system, there is understood to be a terminating mark, called a decimal point, at the right-hand end of the number. Although the decimal point is seldom shown except in numbers involving decimal fractions (covered in chapter 5 of this course), its location must be known. The placement of the decimal point is automatically taken care of when the end 0's are correctly placed.

Practice problems. Multiply in each of the following problems:

1. 287 x 8
2. 67 x 49
3. 940 x 20
4. 807 x 28
5. 694 x 80
6. 9,241 x 7,800

Answers:

1. 2,296
2. 3,283
3. 18,800
4. 22,596
5. 55,520
6. 72,079,800

DIVISION METHODS

Just as multiplication can be considered as repeated addition, division can be considered as repeated subtraction. For example, if we wish to divide 12 by 4 we may subtract 4 from 12 in successive steps and tally the number of times that the subtraction is performed, as follows:

$$\begin{array}{rl} 12 & \\ \underline{4} & * \\ 8 & \\ \underline{4} & * \\ 4 & \\ \underline{4} & * \\ 0 & \end{array}$$

As indicated by the asterisks used as tally marks, 4 has been subtracted 3 times. This result is sometimes described by saying that "4 is contained in 12 three times."

Since successive subtraction is too cumbersome for rapid, concise calculation, methods which treat division as the inverse of multiplication are more useful. Knowledge of the multiplication tables should lead us to an answer for a problem such as 12 ÷ 4 immediately, since we know that 3 x 4 is 12. However, a problem such as 84 ÷ 4 is not so easy to solve by direct reference to the multiplication table.

One way to divide 84 by 4 is to note that 84 is the same as 80 plus 4. Thus 84 ÷ 4 is the same as 80 ÷ 4 plus 4 ÷ 4. In symbols, this can be indicated as follows:

$$\begin{array}{r} 20 + 1 \\ 4\overline{)80 + 4} \end{array}$$

(When this type of division symbol is used, the quotient is written above the vinculum as shown.) Thus, 84 divided by 4 is 21.

From the foregoing example, it can be seen that the regrouping is useful in division as well as in multiplication. However, the mechanical procedure used in division does not include writing down the regrouped form of the dividend. After becoming familiar with the process, we find that the division can be performed directly, one digit at a time, with the regrouping taking place mentally. The following example illustrates this:

$$\begin{array}{r} 14 \\ 4\overline{)56} \\ \underline{4} \\ 16 \\ \underline{16} \end{array}$$

The thought process is as follows: "4 is contained in 5 once" (write 1 in tens place over the 5); "one times 4 is 4" (write 4 in tens place under 5, take the difference, and bring down 6); and "4 is contained in 16 four times" (write 4 in units place over the 6). After a little practice, many people can do the work shown under the dividend mentally and write only the quotient, if the divisor has only 1 digit.

The divisor is sometimes too large to be contained in the first digit of the dividend. The following example illustrates a problem of this kind:

$$\begin{array}{r} 36 \\ 7\overline{)252} \\ \underline{21} \\ 42 \\ \underline{42} \end{array}$$

Since 2 is not large enough to contain 7, we divide 7 into the number formed by the first two digits, 25. Seven is contained 3 times in 25; we write 3 above the 5 of the dividend. Multiplying, 3 times 7 is 21; we write 21 below the first two digits of the dividend. Subtracting, 25 minus 21 is 4; we write down the 4 and bring down the 2 in the units place of the dividend. We have now formed a new dividend, 42. Seven is contained 6 times in 42; we write 6 above the 2 of the dividend. Multiplying as before, 6 times 7 is 42; we write this product below the dividend 42. Subtracting, we have nothing left and the division is complete.

Estimation

When there are two or more digits in the divisor, it is not always easy to determine the first digit of the quotient. An estimate must be made, and the resulting trial quotient may be too large or too small. For example, if 1,862 is to be divided by 38, we might estimate that 38 is contained 5 times in 186 and the first digit of our trial divisor would be 5. However, multiplication reveals that the product of 5 and 38 is larger than 186. Thus we would change the 5 in our quotient to 4, and the problem would then appear as follows:

$$\begin{array}{r} 49 \\ 38\overline{)1862} \\ \underline{152} \\ 342 \\ \underline{342} \end{array}$$

On the other hand, suppose that we had estimated that 38 is contained in 186 only 3 times. We would then have the following:

$$\begin{array}{r} 3 \\ 38\overline{)1862} \\ \underline{114} \\ 72 \end{array}$$

Now, before we make any further moves in the division process, it should be obvious that something is wrong. If our new dividend is large enough to contain the divisor before bringing down a digit from the original dividend, then the trial quotient should have been larger. In other words, our estimate is too small.

Proficiency in estimating trial quotients is gained through practice and familiarity with number combinations. For example, after a little experience we realize that a close estimate can be made in the foregoing problem by thinking of 38 as "almost 40." It is easy to see that 40 is contained 4 times in 186, since 4 times 40 is 160. Also, since 5 times 40 is 200, we are reasonably certain that 5 is too large for our trial divisor.

Uneven Division

In some division problems such as $7 \div 3$, there is no other whole number that, when multiplied by the divisor, will give the dividend. We use the distributive idea to show how division is done in such a case. For example, $7 \div 3$ could be written as follows:

$$\frac{(6 + 1)}{3} = \frac{6}{3} + \frac{1}{3} = 2\,\frac{1}{3}$$

Thus, we see that the quotient also carries one unit that is to be divided by 3. It should now be clear that $3\overline{)37} = 3\overline{)30 + 7}$, and that this can be further reduced as follows:

$$\frac{30}{3} + \frac{6}{3} + \frac{1}{3} = 10 + 2 + \frac{1}{3} = 12\,\frac{1}{3}$$

In elementary arithmetic the part of the dividend that cannot be divided evenly by the divisor is often called a REMAINDER and is placed next to the quotient with the prefix R. Thus, in the foregoing example where the quotient was $12\,\frac{1}{3}$, the quotient could be written 12 R 1. This method of indicating uneven division is useful in examples such as the following:

Suppose that \$13 is available for the purchase of spare parts, and the parts needed cost \$3 each. Four parts can be bought with the available money, and \$1 will be left over. Since it is not possible to buy 1/3 of a part, expressing the result as 4 R 1 gives a more meaningful answer than 4 1/3.

Placement of Decimal Points

In division, as in multiplication, the placement of the decimal point is important. Determining the location of the decimal point and the number of places in the quotient can be relatively simple if the work is kept in the proper columns. For example, notice the vertical alinement in the following problem:

```
     311
31/9,641
   9 3
     34
     31
      31
      31
```

We notice that the first two places in the dividend are used to obtain the first place in the quotient. Since 3 is in the hundreds column there are two more places in the quotient (tens place and units place). The decimal point in the quotient is understood to be directly above the position of the decimal point in the dividend. In the example shown here, the decimal point is not shown but is understood to be immediately after the second 1.

Checking Accuracy

The accuracy of a division of numbers can be checked by multiplying the quotient by the divisor and adding the remainder, if any. The result should equal the dividend. Consider the following example:

```
      5203
  42/218541     Check:  5203
     210                x 42
       85              10406
       84             20812
        141           218526
        126             + 15
         15           218541
```

DENOMINATE NUMBERS

We have learned that denominate numbers are not difficult to add and subtract, provided that units, tens, hundreds, etc., are retained in their respective columns. Multiplication and division of denominate numbers may also be performed with comparative ease, by using the experience gained in addition and subtraction.

Multiplication

In multiplying denominate numbers by integers, no new ideas are needed. If in the problem 3(5 yd 2 ft 6 in.) we remember that we can multiply each part separately to get the correct product (as in the example, 6(8) = 6(3) + 6(5)), we can easily find the product, as follows:

```
  5 yd  2 ft   6 in.
            x 3
 15 yd  6 ft  18 in.
```

Simplifying, this is

17 yd 1 ft 6 in.

When one denominate number is multiplied by another, a question arises concerning the products of the units of measurement. The product of one unit times another of the same kind is one square unit. For example, 1 ft times 1 ft is 1 square foot, abbreviated sq ft; 2 in. times 3 in. is 6 sq in.; etc. If it becomes necessary to multiply such numbers as 2 yd 1 ft times 6 yd 2 ft, the foot units may be converted to fractions of a yard, as follows:

(2 yd 1 ft)(6 yd 2 ft) = (2 1/3 yd)(6 2/3 yd)

In order to complete the multiplication, a knowledge of fractions is needed. Fractions are discussed in chapter 4 of this training course.

Division

The division of denominate numbers requires division of the highest units first; and if there is a remainder, conversion to the next lower unit, and repeated division until all units have been divided.

In the example (24 gal 1 qt 1 pt) ÷ 5, we perform the following steps:

Step 1:

$$\begin{array}{r} 4 \text{ gal} \\ 5\overline{)24 \text{ gal}} \\ \underline{20} \\ 4 \text{ gal (left over)} \end{array}$$

Step 2: Convert the 4 gal left over to 16 qt and add to the 1 qt.

Step 3:

$$\begin{array}{r} 3 \text{ qt} \\ 5\overline{)17 \text{ qt}} \\ \underline{15} \\ 2 \text{ qt (left over)} \end{array}$$

Step 4: Convert the 2 qt left over to 4 pt and add to the 1 pt.

Step 5:

$$\begin{array}{r} 1 \text{ pt} \\ 5\overline{)5 \text{ pt}} \end{array}$$

Therefore, 24 gal 1 qt 1 pt divided by 5 is 4 gal 3 qt 1 pt.

Practice problems. In problems 1 through 4, divide as indicated. In problems 5 through 8, multiply or divide as indicated.

1. $549 \div 9$

2. 470/63

3. $25\overline{)2{,}300}$

4. $64\overline{)74{,}816}$

5. $\begin{array}{r} 4 \text{ hr } 26 \text{ min } 16 \text{ sec} \\ \underline{\times\ 5} \end{array}$

6. 3(4 gal 3 qt 1 pt)

7. $\dfrac{67 \text{ deg } 43 \text{ min } 12 \text{ sec}}{2}$

8. $5\overline{)63 \text{ lb } 11 \text{ oz}}$

Answers:

1. 61

2. 7 R 29

3. 92

4. 1,169

5. 22 hr 11 min 20 sec

6. 14 gal 2 qt 1 pt

7. 33 deg 51 min 36 sec

8. 12 lb 11 4/5 oz

ORDER OF OPERATIONS

When a series of operations involving addition, subtraction, multiplication, or division is indicated, the order in which the operations are performed is important only if division is involved or if the operations are mixed. A series of individual additions, subtractions, or multiplications may be performed in any order. Thus, in

$$4 + 2 + 7 + 5 = 18$$

or

$$100 - 20 - 10 - 3 = 67$$

or

$$4 \times 2 \times 7 \times 5 = 280$$

the numbers may be combined in any order desired. For example, they may be grouped easily to give

$$6 + 12 = 18$$

and

$$97 - 30 = 67$$

and

$$40 \times 7 = 280$$

A series of divisions should be taken in the order written.

Thus,

$$100 \div 10 \div 2 = 10 \div 2 = 5$$

In a series of mixed operations, perform multiplications first, division next, and finally additions and subtractions.

For example

$$100 \div 4 \times 5 = 100 \div 20 = 5$$

and

$$60 - 25 \div 5 = 60 - 5 = 55$$

Now consider

$$\begin{aligned} & 60 - 25 \div 5 + 15 - 100 + 4 \times 10 \\ &= 60 - 25 \div 5 + 15 - 100 + 40 \\ &= 60 - 5 + 15 - 100 + 40 \\ &= 115 - 105 \\ &= 10 \end{aligned}$$

Notice that $25 \div 5$ could be evaluated at the same time that 4×10 is evaluated, since no other multiplication is to be performed in the first part of the problem.

Practice problems. Evaluate each of the following expressions:

1. $9 \div 3 + 2$
2. $18 - 2 \times 5 + 4$
3. $90 \div 2 \div 9$
4. $75 \div 5 \times 3 \div 5$
5. $7 + 1 - 8 \times 4 \div 16$

Answers:

1. 5
2. 12
3. 5
4. 1
5. 6

MULTIPLES AND FACTORS

Any number that is exactly divisible by a given number is a MULTIPLE of the given number. For example, 24 is a multiple of 2, 3, 4, 6, 8, and 12, since it is divisible by each of these numbers. Saying that 24 is a multiple of 3, for instance, is equivalent to saying that 3 multiplied by some whole number will give 24. Any number is a multiple of itself and also of 1.

Any number that is a multiple of 2 is an EVEN NUMBER. The even numbers begin with 2 and progress by 2's as follows:

$$2, 4, 6, 8, 10, 12, \ldots$$

Any number that is not a multiple of 2 is an ODD NUMBER. The odd numbers begin with 1 and progress by 2's, as follows:

$$1, 3, 5, 7, 9, 11, 13, \ldots$$

Any number that can be divided into a given number without a remainder is a FACTOR of the given number. The given number is a multiple of any number that is one of its factors. For example, 2, 3, 4, 6, 8, and 12 are factors of 24. The following four equalities show various combinations of the factors of 24:

$$24 = 24 \cdot 1 \qquad 24 = 8 \cdot 3$$
$$24 = 12 \cdot 2 \qquad 24 = 6 \cdot 4$$

If the number 24 is factored as completely as possible, it assumes the form

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$

ZERO AS A FACTOR

If any number is multipiied by zero, the product is zero. For example, 5 times zero equals zero and may be written $5(0) = 0$. The zero factor law tells us that, if the product of two or more factors is zero, at least one of the factors must be zero.

PRIME FACTORS

A number that has factors other than itself and 1 is a COMPOSITE NUMBER. For example, the number 15 is composite. It has the factors 5 and 3.

A number that has no factors except itself and 1 is a PRIME NUMBER. Since it is sometimes advantageous to separate a composite number into prime factors, it is helpful to be able to recognize a few prime numbers quickly. The following series shows all the prime numbers up to 60:

1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59.

Notice that 2 is the only even prime number. All other even numbers are divisible by 2. Notice also that 51, for example, does not appear in the series, since it is a composite number equal to 3 x 17.

If a factor of a number is prime, it is called a PRIME FACTOR. To separate a number into prime factors, begin by taking out the smallest factor. If the number is even, take out all the 2's first, then try 3 as a factor, etc. Thus, we have the following example:

$$\begin{aligned} 540 &= 2 \cdot 270 \\ &= 2 \cdot 2 \cdot 135 \\ &= 2 \cdot 2 \cdot 3 \cdot 45 \\ &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 15 \\ &= 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \end{aligned}$$

Since 1 is an understood factor of every number, we do not waste space recording it as one of the factors in a presentation of this kind.

A convenient way of keeping track of the prime factors is in the short division process as follows:

$$\begin{array}{l} 2\underline{/540} \\ 2\underline{/270} \\ 3\underline{/135} \\ 3\underline{/45\ } \\ 3\underline{/15\ } \\ 5\underline{/5\ \ } \\ \quad 1 \end{array}$$

If a number is odd, its factors will be odd numbers. To separate an odd number into prime factors, take out the 3's first, if there are any. Then try 5 as a factor, etc. As an example,

$$\begin{aligned} 5{,}775 &= 3 \cdot 1{,}925 \\ &= 3 \cdot 5 \cdot 385 \\ &= 3 \cdot 5 \cdot 5 \cdot 77 \\ &= 3 \cdot 5 \cdot 5 \cdot 7 \cdot 11 \end{aligned}$$

Practice problems:

1. Which of the following are prime numbers and which are composite numbers?

   25, 7, 18, 29, 51

2. What prime numbers are factors of 36?
3. Which of the following are multiples of 3?

   45, 53, 51, 39, 47

4. Find the prime factors of 27.

Answers:

1. Prime: 7, 29
   Composite: 25, 18, 51
2. $36 = 2 \cdot 2 \cdot 3 \cdot 3$
3. 45, 51, 39
4. $27 = 3 \cdot 3 \cdot 3$

Tests for Divisibility

It is often useful to be able to tell by inspection whether a number is exactly divisible by one or more of the digits from 2 through 9. An expression which is frequently used, although it is sometimes misleading, is "evenly divisible." This expression has nothing to do with the concept of even and odd numbers, and it probably should be avoided in favor of the more descriptive expression, "exactly divisible." For the remainder of this discussion, the word "divisible" has the same meaning as "exactly divisible." Several tests for divisibility are listed in the following paragraphs:

1. A number is divisible by 2 if its right-hand digit is even.

2. A number is divisible by 3 if the sum of its digits is divisible by 3. For example, the digits of the number 6,561 add to produce the sum 18. Since 18 is divisible by 3, we know that 6,561 is divisible by 3.

3. A number is divisible by 4 if the number formed by the two right-hand digits is divisible by 4. For example, the two right-hand digits of the number 3,524 form the number 24. Since 24 is divisible by 4, we know that 3,524 is divisible by 4.

4. A number is divisible by 5 if its right-hand digit is 0 or 5.

5. A number is divisible by 6 if it is even and the sum of its digits is divisible by 3. For example, the sum of the digits of 64,236 is 21, which is divisible by 3. Since 64,236 is also an even number, we know that it is divisible by 6.

6. No short method has been found for determining whether a number is divisible by 7.

7. A number is divisible by 8 if the number formed by the three right-hand digits is divisible by 8. For example, the three right-hand digits of the number 54,272 form the number 272, which is divisible by 8. Therefore, we know that 54,272 is divisible by 8.

8. A number is divisible by 9 if the sum of its digits is divisible by 9. For example, the sum of the digits of 546,372 is 27, which is divisible by 9. Therefore we know that 546,372 is divisible by 9.

Practice problems. Check each of the following numbers for divisibility by all of the digits except 7:

1. 242,431,231,320
2. 844,624,221,840
3. 988,446,662,640
4. 207,634,542,480

Answers: All of these numbers are divisible by 2, 3, 4, 5, 6, 8, and 9.

CHAPTER 3

SIGNED NUMBERS

The positive numbers with which we have worked in previous chapters are not sufficient for every situation which may arise. For example, a negative number results in the operation of subtraction when the subtrahend is larger than the minuend.

NEGATIVE NUMBERS

When the subtrahend happens to be larger than the minuend, this fact is indicated by placing a minus sign in front of the difference, as in the following:

$$12 - 20 = -8$$

The difference, -8, is said to be NEGATIVE. A number preceded by a minus sign is a NEGATIVE NUMBER. The number -8 is read "minus eight." Such a number might arise when we speak of temperature changes. If the temperature was 12 degrees yesterday and dropped 20 degrees today, the reading today would be 12 - 20, or -8 degrees.

Numbers that show either a plus or minus sign are called SIGNED NUMBERS. An unsigned number is understood to be positive and is treated as though there were a plus sign preceding it.

If it is desired to emphasize the fact that a number is positive, a plus sign is placed in front of the number, as in +5, which is read "plus five." Therefore, either +5 or 5 indicates that the number 5 is positive. If a number is negative, a minus sign must appear in front of it, as in -9.

In dealing with signed numbers it should be emphasized that the plus and minus signs have two separate and distinct functions. They may indicate whether a number is positive or negative, or they may indicate the operation of addition or subtraction.

When operating entirely with positive numbers, it is not necessary to be concerned with this distinction since plus or minus signs indicate only addition or subtraction. However, when negative numbers are also involved in a computation, it is important to distinguish between a sign of operation and the sign of a number.

DIRECTION OF MEASUREMENT

Signed numbers provide a convenient way of indicating opposite directions with a minimum of words. For example, an altitude of 20 ft above sea level could be designated as +20 ft. The same distance below sea level would then be designated as -20 ft. One of the most common devices utilizing signed numbers to indicate direction of measurement is the thermometer.

Thermometer

The Celsius (centigrade) thermometer shown in figure 3-1 illustrates the use of positive and negative numbers to indicate direction of travel above and below 0. The 0 mark is the changeover point, at which the signs of the scale numbers change from - to +.

When the thermometer is heated by the surrounding air or by a hot liquid in which it is placed, the mercury expands and travels up the tube. After the expanding mercury passes 0, the mark at which it comes to rest is read as a positive temperature. If the thermometer is allowed to cool, the mercury contracts. After passing 0 in its downward movement, any mark at which it comes to rest is read as a negative temperature.

Rectangular Coordinate System

As a matter of convenience, mathematicians have agreed to follow certain conventions as to the use of signed numbers in directional measurement. For example, in figure 3-2, a direction to the right along the horizontal line is positive, while the opposite direction (toward the left) is negative. On the vertical line, direction upward is positive, while direction downward is negative. A distance of -3 units along the horizontal line indicates a measurement of 3 units to the left of starting point 0. A distance of -3 units on the vertical line indicates

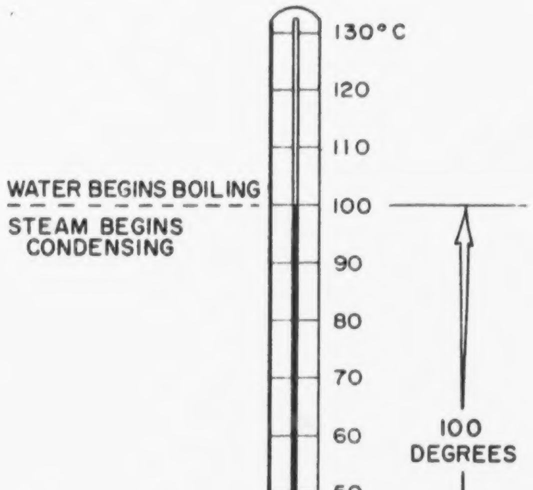

130°C
120
110
100
90
80
70
60
50
40
30
20
+10
0
-10
-20
-30
-40
WATER BEGINS BOILING
STEAM BEGINS CONDENSING
100 DEGREES
ICE BEGINS MELTING
WATER BEGINS FREEZING
MERCURY
BOILING WATER

Figure 3-1.—Celsius (centigrade) temperature scale.

a measurement of 3 units below the starting point.

The two lines of the rectangular coordinate system which pass through the 0 position are the vertical axis and horizontal axis. Other vertical and horizontal lines may be included, forming a grid. When such a grid is used for the location of points and lines, the resulting "picture" containing points and lines is called a GRAPH.

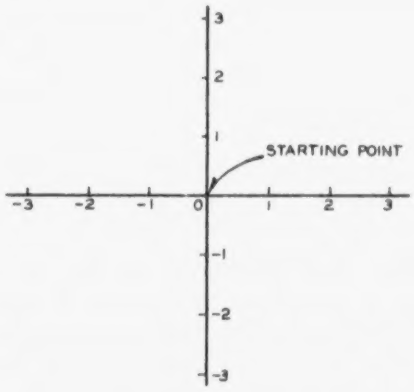

3
2
1
STARTING POINT
-3
-2
-1
0
1
2
3
-1
-2
-3

Figure 3-2.—Rectangular coordinate system.

The Number Line

Sometimes it is important to know the relative greatness (magnitude) of positive and negative numbers. To determine whether a particular number is greater or less than another number, think of all the numbers both positive and negative as being arranged along a horizontal line. (See fig. 3-3.)

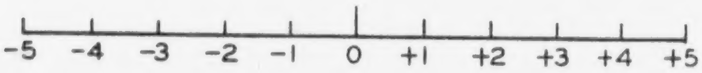

-5
-4
-3
-2
-1
0
+1
+2
+3
+4
+5

Figure 3-3.—Number line showing both positive and negative numbers.

Place zero at the middle of the line. Let the positive numbers extend from zero toward the right. Let the negative numbers extend from zero toward the left. With this arrangement, positive and negative numbers are so located that they progress from smaller to larger numbers as we move from left to right along the line. Any number that lies to the right of a given number is greater than the given number. A number that lies to the left of a given number is less than the given number. This arrangement shows that any negative number is smaller than any positive number.

The symbol for "greater than" is $>$. The symbol for "less than" is $<$. It is easy to distinguish between these symbols because the symbol used always opens toward the larger number. For example, "7 is greater than 4" can be written $7 > 4$ and "-5 is less than -1" can be written $-5 < -1$.

Absolute Value

The ABSOLUTE VALUE of a number is its numerical value when the sign is dropped. The absolute value of either +5 or -5 is 5. Thus, two numbers that differ only in sign have the same absolute value.

The symbol for absolute value consists of two vertical bars placed one on each side of the number, as in $|-5| = 5$. Consider also the following:

$$|4 - 20| = 16$$
$$|+7| = |-7| = 7$$

The expression $|-7|$ is read "absolute value of minus seven."

When positive and negative numbers are used to indicate direction of measurement, we are concerned only with absolute value, if we wish to know only the distance covered. For example, in figure 3-2, if an object moves to the left from the starting point to the point indicated by -2, the actual distance covered is 2 units. We are concerned only with the fact that $|-2| = 2$, if our only interest is in the distance and not the direction.

OPERATING WITH SIGNED NUMBERS

The number line can be used to demonstrate addition of signed numbers. Two cases must be considered; namely, adding numbers with like signs and adding numbers with unlike signs.

ADDING WITH LIKE SIGNS

As an example of addition with like signs, suppose that we use the number line (fig. 3-4) to add 2 + 3. Since these are signed numbers, we indicate this addition as (+2) + (+3). This emphasizes that, among the three + signs shown, two are number signs and one is a sign of operation. Line a (fig. 3-4) above the number line shows this addition. Find 2 on the number line. To add 3 to it, go three units more in a positive direction and get 5.

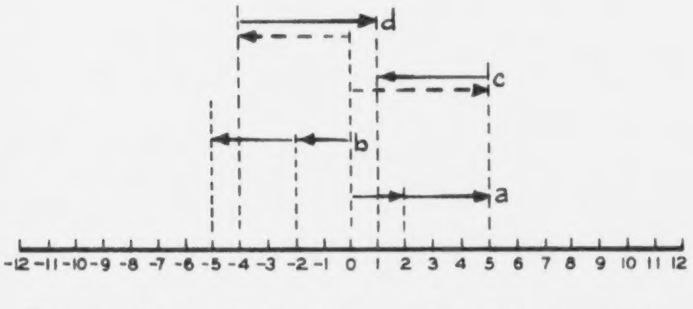


Figure 3-4.—Using the number line to add.

To add two negative numbers on the number line, such as -2 and -3, find -2 on the number line and then go three units more in the negative direction to get -5, as in b (fig. 3-4) above the number line.

Observation of the results of the foregoing operations on the number line leads us to the following conclusion, which may be stated as a law: To add numbers with like signs, add the absolute values and prefix the common sign.

ADDING WITH UNLIKE SIGNS

To add a positive and a negative number, such as (-4) + (+5), find +5 on the number line and go four units in a negative direction, as in line c above the number line in figure 3-4. Notice that this addition could be performed in the other direction. That is, we could start at -4 and move 5 units in the positive direction. (See line d, fig. 3-4.)

The results of our operations with mixed signs on the number line lead to the following conclusion, which may be stated as a law: To add numbers with unlike signs, find the difference between their absolute values and prefix the sign of the numerically greater number.

The following examples show the addition of the numbers 3 and 5 with the four possible combinations of signs:

$$\begin{array}{cccc} 3 & -3 & 3 & -3 \\ 5 & -5 & -5 & 5 \\ \hline 8 & -8 & -2 & 2 \end{array}$$

In the first example, 3 and 5 have like signs and the common sign is understood to be positive. The sum of the absolute values is 8 and no sign is prefixed to this sum, thus signifying that the sign of the 8 is understood to be positive.

In the second example, the 3 and 5 again have like signs, but their common sign is negative. The sum of the absolute values is 8, and this time the common sign is prefixed to the sum. The answer is thus -8.

In the third example, the 3 and 5 have unlike signs. The difference between their absolute values is 2, and the sign of the larger addend is negative. Therefore, the answer is -2.

In the fourth example, the 3 and 5 again have unlike signs. The difference of the absolute values is still 2, but this time the sign of the larger addend is positive. Therefore, the sign prefixed to the 2 is positive (understood) and the final answer is simply 2.

These four examples could be written in a different form, emphasizing the distinction between the sign of a number and an operational sign, as follows:

$$\begin{aligned}(+3) + (+5) &= +8\\(-3) + (-5) &= -8\\(+3) + (-5) &= -2\\(-3) + (+5) &= +2\end{aligned}$$

Practice problems. Add as indicated:

1. $-10 + 5 = (-10) + (+5) = ?$
2. Add -9, -16, and 25
3. $- 7 - 1 - 3 = (-7) + (-1) + (-3) = ?$
4. Add -22 and -13

Answers:

1. -5	3. -11
2. 0	4. -35

SUBTRACTION

Subtraction is the inverse of addition. When subtraction is performed, we "take away" the subtrahend. This means that whatever the value of the subtrahend, its effect is to be reversed when subtraction is indicated. In addition, the sum of 5 and -2 is 3. In subtraction, however, to take away the effect of the -2, the quantity +2 must be added. Thus the difference between +5 and -2 is +7.

Keeping this idea in mind, we may now proceed to examine the various combinations of subtraction involving signed numbers. Let us first consider the four possibilities where the minuend is numerically greater than the subtrahend, as in the following examples:

$$\begin{array}{r}8\\5\\\hline 3\end{array}\qquad\begin{array}{r}8\\-5\\\hline 13\end{array}\qquad\begin{array}{r}-8\\5\\\hline -13\end{array}\qquad\begin{array}{r}-8\\-5\\\hline -3\end{array}$$

We may show how each of these results is obtained by use of the number line, as shown in figure 3-5.

In the first example, we find +8 on the number line, then subtract 5 by making a movement that reverses its sign. Thus, we move to the left 5 units. The result (difference) is +3. (See line a, fig. 3-5.)

In the second example, we find +8 on the number line, then subtract (-5) by making a movement that will reverse its sign. Thus we move to the right 5 units. The result in this case is +13. (See line b, fig. 3-5.)

In the third example, we find -8 on the number line, then subtract 5 by making a movement that reverses its sign. Thus we move to the left 5 units. The result is -13. (See line c, fig. 3-5.)

In the fourth example, we find -8 on the number line, then reverse the sign of -5 by moving 5 units to the right. The result is -3. (See line d, fig. 3-5.)

Next, let us consider the four possibilities that arise when the subtrahend is numerically greater than the minuend, as in the following examples:

$$\begin{array}{r}5\\8\\\hline -3\end{array}\qquad\begin{array}{r}5\\-8\\\hline 13\end{array}\qquad\begin{array}{r}-5\\8\\\hline -13\end{array}\qquad\begin{array}{r}-5\\-8\\\hline 3\end{array}$$

In the first example, we find +5 on the number line, then subtract 8 by making a movement

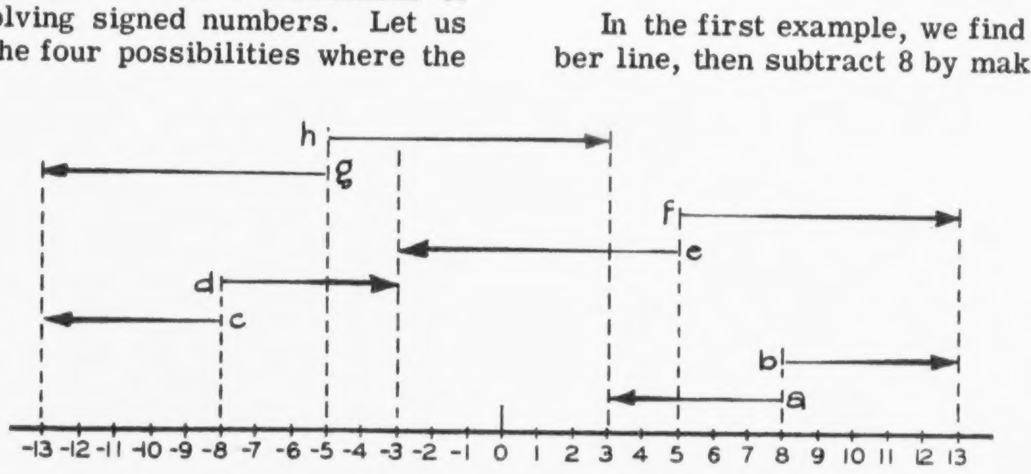


Figure 3-5.—Subtraction by use of the number line.

that reverses its sign. Thus we move to the left 8 units. The result is -3. (See line e, fig. 3-5.)

In the second example, we find +5 on the number line, then subtract -8 by making a movement to the right that reverses its sign. The result is 13. (See line f, fig. 3-5.)

In the third example, we find -5 on the number line, then reverse the sign of 8 by a movement to the left. The result is -13. (See line g, fig. 3-5.)

In the fourth example, we find -5 on the number line, then reverse the sign of -8 by a movement to the right. The result is 3. (See line h, fig. 3-5.)

Careful study of the preceding examples leads to the following conclusion, which is stated as a law for subtraction of signed numbers: In any subtraction problem, mentally change the sign of the subtrahend and proceed as in addition.

Practice problems. In problems 1 through 4, subtract the lower number from the upper. In 5 through 8, subtract as indicated.

1. 17	2. -12	3. -9	4. 7
-10	8	-13	16

5. 1 -(-5) = ?
6. -6 -(-8) = ?
7. 14 - 7 -(-3) = ?
8. -9 - 2 = ?

Answers:

1. 27	2. -20	3. 4	4. -9
5. 6	6. 2	7. 10	8. -11

MULTIPLICATION

To explain the rules for multiplication of signed numbers, we recall that multiplication of whole numbers may be thought of as shortened addition. Two types of multiplication problems must be examined; the first type involves numbers with unlike signs, and the second involves numbers with like signs.

Unlike Signs

Consider the example 3(-4), in which the multiplicand is negative. This means we are to add -4 three times; that is, 3(-4) is equal to (-4) + (-4) + (-4), which is equal to -12. For example, if we have three 4-dollar debts, we owe 12 dollars in all.

When the multiplier is negative, as in -3(7), we are to take away 7 three times. Thus, -3(7) is equal to -(7) - (7) - (7) which is equal to -21. For example, if 7 shells were expended in one firing, 7 the next, and 7 the next, there would be a loss of 21 shells in all. Thus, the rule is as follows: The product of two numbers with unlike signs is negative.

The law of signs for unlike signs is sometimes stated as follows: Minus times plus is minus; plus times minus is minus. Thus a problem such as 3(-4) can be reduced to the following two steps:

1. Multiply the signs and write down the sign of the answer before working with the numbers themselves.
2. Multiply the numbers as if they were unsigned numbers.

Using the suggested procedure, the sign of the answer for 3(-4) is found to be minus. The product of 3 and 4 is 12, and the final answer is -12. When there are more than two numbers to be multiplied, the signs are taken in pairs until the final sign is determined.

Like Signs

When both factors are positive, as in 4(5), the sign of the product is positive. We are to add +5 four times, as follows:

$$4(5) = 5 + 5 + 5 + 5 = 20$$

When both factors are negative, as in -4(-5), the sign of the product is positive. We are to take away -5 four times.

$$\begin{aligned} -4(-5) &= -(-5) - (-5) - (-5) - (-5) \\ &= +5 +5 +5 +5 \\ &= 20 \end{aligned}$$

Remember that taking away a negative 5 is the same as adding a positive 5. For example, suppose someone owes a man 20 dollars and pays him back (or diminishes the debt) 5 dollars at a time. He takes away a debt of 20 dollars by giving him four positive 5-dollar bills, or a total of 20 positive dollars in all.

The rule developed by the foregoing example is as follows: The product of two numbers with like signs is positive.

Knowing that the product of two positive numbers or two negative numbers is positive, we can conclude that the product of any even number of negative numbers is positive. Similarly, the product of any odd number of negative numbers is negative.

The laws of signs may be combined as follows: Minus times plus is minus; plus times minus is minus; minus times minus is plus; plus times plus is plus. Use of this combined rule may be illustrated as follows:

$$4(-2) \cdot (-5) \cdot (6) \cdot (-3) = -720$$

Taking the signs in pairs, the understood plus on the 4 times the minus on the 2 produces a minus. This minus times the minus on the 5 produces a plus. This plus times the understood plus on the 6 produces a plus. This plus times the minus on the 3 produces a minus, so we know that the final answer is negative. The product of the numbers, disregarding their signs, is 720; therefore, the final answer is -720.

Practice problems. Multiply as indicated:

1. $5(-8) = ?$
2. $-7(3)(2) = ?$
3. $6(-1)(-4) = ?$
4. $-2(3)(-4)(5)(-6) = ?$

Answers:

1. -40
2. -42
3. 24
4. -720

DIVISION

Because division is the inverse of multiplication, we can quickly develop the rules for division of signed numbers by comparison with the corresponding multiplication rules, as in the following examples:

1. Division involving two numbers with unlike signs is related to multiplication with unlike signs, as follows:

$$3(-4) = -12$$

Therefore, $\frac{-12}{3} = -4$

Thus, the rule for division with unlike signs is: The quotient of two numbers with unlike signs is negative.

2. Division involving two numbers with like signs is related to multiplication with like signs, as follows:

$$3(-4) = -12$$

Therefore, $\frac{-12}{-4} = 3$

Thus the rule for division with like signs is: The quotient of two numbers with like signs is positive.

The following examples show the application of the rules for dividing signed numbers:

$$\frac{12}{3} = 4 \qquad \frac{-12}{3} = -4$$

$$\frac{-12}{-3} = 4 \qquad \frac{12}{-3} = -4$$

Practice problems. Multiply and divide as indicated:

1. $15 \div -5$
2. $-2(-3)/-6$
3. $\frac{(-3)(4)}{-6}$
4. $-81/9$

Answers:

1. -3
2. -1
3. 2
4. -9

SPECIAL CASES

Two special cases arise frequently in which the laws of signs may be used to advantage. The first such usage is in simplifying subtraction; the second is in changing the signs of the numerator and denominator when division is indicated in the form of a fraction.

Subtraction

The rules for subtraction may be simplified by use of the laws of signs, if each expression to be subtracted is considered as being multiplied by a negative sign. For example, $4 -(-5)$ is the same as $4 + 5$, since minus times minus is plus. This result also establishes a basis for the rule governing removal of parentheses.

The parentheses rule, as usually stated, is: Parentheses preceded by a minus sign may be removed, if the signs of all terms within the parentheses are changed. This is illustrated as follows:

$$12 -(3 - 2 + 4) = 12 - 3 + 2 - 4$$

The reason for the changes of sign is clear when the negative sign preceding the parentheses is considered to be a multiplier for the whole parenthetical expression.

Division in Fractional Form

Division is often indicated by writing the dividend as the numerator, and the divisor as the denominator, of a fraction. In algebra, every fraction is considered to have three signs. The numerator has a sign, the denominator has a sign, and the fraction itself, taken as a whole, has a sign. In many cases, one or more of these signs will be positive, and thus will not be shown. For example, in the following fraction the sign of the numerator and the sign of the denominator are both positive (understood) and the sign of the fraction itself is negative:

$$-\frac{4}{5}$$

Fractions with more than one negative sign are always reducible to a simpler form with at most one negative sign. For example, the sign of the numerator and the sign of the denominator may be both negative. We note that minus divided by minus gives the same result as plus divided by plus. Therefore, we may change to the less complicated form having plus signs (understood) for both numerator and denominator, as follows:

$$\frac{-15}{-5} = \frac{+15}{+5} = \frac{15}{5}$$

Since -15 divided by -5 is 3, and 15 divided by 5 is also 3, we conclude that the change of sign does not alter the final answer. The same reasoning may be applied in the following example, in which the sign of the fraction itself is negative:

$$-\frac{-15}{-5} = -\frac{+15}{+5} = -\frac{15}{5}$$

When the fraction itself has a negative sign, as in this example, the fraction may be enclosed in parentheses temporarily, for the purpose of working with the numerator and denominator only. Then the sign of the fraction is applied separately to the result, as follows:

$$-\frac{-15}{-5} = -\left(\frac{-15}{-5}\right) = -(3) = -3$$

All of this can be done mentally.

If a fraction has a negative sign in one of the three sign positions, this sign may be moved to another position. Such an adjustment is an advantage in some types of complicated expressions involving fractions. Examples of this type of sign change follow:

$$-\frac{15}{5} = \frac{-15}{5} = \frac{15}{-5}$$

In the first expression of the foregoing example, the sign of the numerator is positive (understood) and the sign of the fraction is negative. Changing both of these signs, we obtain the second expression. To obtain the third expression from the second, we change the sign of the numerator and the sign of the denominator. Observe that the sign changes in each case involve a pair of signs. This leads to the law of signs for fractions: Any two of the three signs of a fraction may be changed without altering the value of the fraction.

AXIOMS AND LAWS

An axiom is a self-evident truth. It is a truth that is so universally accepted that it does not require proof. For example, the statement that "a straight line is the shortest distance between two points" is an axiom from plane geometry. One tends to accept the truth of an axiom without proof, because anything which is axiomatic is, by its very nature, obviously true. On the other hand, a law (in the mathematical sense) is the result of defining certain quantities and relationships and then developing logical conclusions from the definitions.

AXIOMS OF EQUALITY

The four axioms of equality with which we are concerned in arithmetic and algebra are stated as follows:

1. If the same quantity is added to each of two equal quantities, the resulting quantities are equal. This is sometimes stated as follows: If equals are added to equals, the results are equal. For example, by adding the same quantity (3) to both sides of the following equation, we obtain two sums which are equal:

$$\begin{aligned} -2 &= -3 + 1 \\ -2 + 3 &= -3 + 1 + 3 \\ 1 &= 1 \end{aligned}$$

2. If the same quantity is subtracted from each of two equal quantities, the resulting quantities are equal. This is sometimes stated as follows: If equals are subtracted from equals, the results are equal. For example, by subtracting 2 from both sides of the following equation we obtain results which are equal:

$$\begin{aligned} 5 &= 2 + 3 \\ 5 - 2 &= 2 + 3 - 2 \\ 3 &= 3 \end{aligned}$$

3. If two equal quantities are multiplied by the same quantity, the resulting products are equal. This is sometimes stated as follows: If equals are multiplied by equals, the products are equal. For example, both sides of the following equation are multiplied by -3 and equal results are obtained:

$$\begin{aligned} 5 &= 2 + 3 \\ (-3)(5) &= (-3)(2 + 3) \\ -15 &= -15 \end{aligned}$$

4. If two equal quantities are divided by the same quantity, the resulting quotients are equal. This is sometimes stated as follows: If equals are divided by equals, the results are equal. For example, both sides of the following equation are divided by 3, and the resulting quotients are equal:

$$12 + 3 = 15$$

$$\frac{12 + 3}{3} = \frac{15}{3}$$

$$4 + 1 = 5$$

These axioms are especially useful when letters are used to represent numbers. If we know that $5x = -30$, for instance, then dividing both $5x$ and -30 by 5 leads to the conclusion that $x = -6$.

LAWS FOR COMBINING NUMBERS

Numbers are combined in accordance with the following basic laws:

1. The associative laws of addition and multiplication.
2. The commutative laws of addition and multiplication.
3. The distributive law.

Associative Law of Addition

The word "associative" suggests association or grouping. This law states that the sum of three or more addends is the same regardless of the manner in which they are grouped. For example, $6 + 3 + 1$ is the same as $6 + (3 + 1)$ or $(6 + 3) + 1$ or $(6 + 1) + 3$.

This law can be applied to subtraction by changing signs in such a way that all negative signs are treated as number signs rather than operational signs. That is, some of the addends can be negative numbers. For example, $6 - 4 - 2$ can be rewritten as $6 + (-4) + (-2)$. By the associative law, this is the same as

$$6 + [(-4) + (-2)] \text{ or } [6 + (-4)] + (-2).$$

However, $6 - 4 - 2$ is not the same as $6 - (4 - 2)$; the terms must be expressed as addends before applying the associative law of addition.

Associative Law of Multiplication

This law states that the product of three or more factors is the same regardless of the manner in which they are grouped. For example, $6 \cdot 3 \cdot 2$ is the same as $(6 \cdot 3) \cdot 2$ or $6 \cdot (3 \cdot 2)$ or $(6 \cdot 2) \cdot 3$. Negative signs require no special treatment in the application of this law. For example, $6 \cdot (-4) \cdot (-2)$ is the same as $[6 \cdot (-4)] \cdot (-2)$ or $6 \cdot [(-4) \cdot (-2)]$.

Commutative Law of Addition

The word "commute" means to change, substitute or move from place to place. The commutative law of addition states that the sum of two or more addends is the same regardless of the order in which they are arranged. For example, $4 + 3 + 2$ is the same as $4 + 2 + 3$ or $2 + 4 + 3$.

This law can be applied to subtraction by changing signs so that all negative signs become number signs and all signs of operation are positive. For example, $5 - 3 - 2$ is changed to $5 + (-3) + (-2)$, which is the same as $5 + (-2) + (-3)$ or $(-3) + 5 + (-2)$.

Commutative Law of Multiplication

This law states that the product of two or more factors is the same regardless of the order in which the factors are arranged. For example, $3 \cdot 4 \cdot 5$ is the same as $5 \cdot 3 \cdot 4$ or

4 · 3 · 5. Negative signs require no special treatment in the application of this law. For example, 2 · (-4) · (-3) is the same as (-4) · (-3) · 2 or (-3) · 2 · (-4).

Distributive Law

This law combines the operations of addition and multiplication. The word "distributive" refers to the distribution of a common multiplier among the terms of an additive expression. For example,

$$\begin{aligned} 2(3 + 4 + 5) &= 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 \\ &= 6 + 8 + 10 \end{aligned}$$

To verify the distributive law, we note that 2(3 + 4 + 5) is the same as 2(12) or 24. Also, 6 + 8 + 10 is 24. For application of the distributive law where negative signs appear, the following procedure is recommended:

$$\begin{aligned} 3(4 - 2) &= 3\ [4 + (-2)] \\ &= 3(4) + 3(-2) \\ &= 12 - 6 \\ &= 6 \end{aligned}$$

CHAPTER 4

COMMON FRACTIONS

The emphasis in previous chapters of this course has been on integers (whole numbers). In this chapter, we turn our attention to numbers which are not integers. The simplest type of number other than an integer is a COMMON FRACTION. Common fractions and integers together comprise a set of numbers called the RATIONAL NUMBERS; this set is a subset of the set of real numbers.

The number line may be used to show the relationship between integers and fractions. For example, if the interval between 0 and 1 is marked off to form three equal spaces (thirds), then each space so formed is one-third of the total interval. If we move along the number line from 0 toward 1, we will have covered two of the three "thirds" when we reach the second mark. Thus the position of the second mark represents the number 2/3. (See fig. 4-1.)

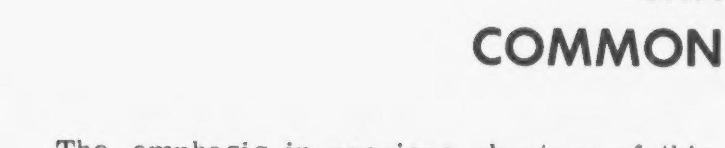


Figure 4-1.—Integers and fractions on the number line.

The numerals 2 and 3 in the fraction 2/3 are named so that we may distinguish between them; 2 is the NUMERATOR and 3 is the DENOMINATOR. In general, the numeral above the dividing line in a fraction is the numerator and the numeral below the line is the denominator. The numerator and denominator are the TERMS of the fraction. The word "numerator" is related to the word "enumerate." To enumerate means to "tell how many"; thus the numerator tells us how many fractional parts we have in the indicated fraction. To denominate means to "give a name" or "tell what kind"; thus the denominator tells us what kind of parts we have (halves, thirds, fourths, etc.).

Attempts to define the word "fraction" in mathematics usually result in a statement similar to the following: A fraction is an indicated division. Any division may be indicated by placing the dividend over the divisor and drawing a line between them. By this definition, any number which can be written as the ratio of two integers (one integer over the other) can be considered as a fraction. This leads to a further definition: Any number which can be expressed as the ratio of two integers is a RATIONAL number. Notice that every integer is a rational number, because we can write any integer as the numerator of a fraction having 1 as its denominator. For example, 5 is the same as 5/1. It should be obvious from the definition that every common fraction is also a rational number.

TYPES OF FRACTIONS

Fractions are often classified as proper or improper. A proper fraction is one in which the numerator is numerically smaller than the denominator. An improper fraction has a numerator which is larger than its denominator.

MIXED NUMBERS

When the denominator of an improper fraction is divided into its numerator, a remainder is produced along with the quotient, unless the numerator happens to be an exact multiple of the denominator. For example, 7/5 is equal to 1 plus a remainder of 2. This remainder may be shown as a dividend with 5 as its divisor, as follows:

$$\frac{7}{5} = \frac{5 + 2}{5} = 1 + \frac{2}{5}$$

The expression 1 + 2/5 is a MIXED NUMBER. Mixed numbers are usually written without showing the plus sign; that is, 1 + 2/5 is the same as $1\frac{2}{5}$ or 1 2/5. When a mixed number is written as 1 2/5, care must be taken to insure that there is a space between the 1 and the 2; otherwise, 1 2/5 might be taken to mean 12/5.

MEASUREMENT FRACTIONS

Measurement fractions occur in problems such as the following:

If \$2 were spent for a stateroom rug at \$3 per yard, how many yards were bought? If \$6 had been spent we could find the number of yards by simply dividing the cost per yard into the amount spent. Since 6/3 is 2, two yards could be bought for \$6. The same reasoning applies when \$2 are spent, but in this case we can only indicate the amount purchased as the indicated division 2/3. Figure 4-2 shows a diagram for both the \$6 purchase and the \$2 purchase.

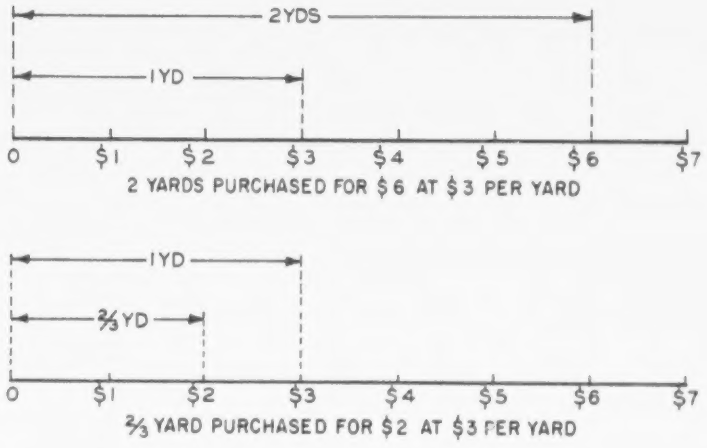

2YDS
1YD
0
$1
$2
$3
$4
$5
$6
$7
2 YARDS PURCHASED FOR $6 AT $3 PER YARD
1YD
⅔YD
⅔ YARD PURCHASED FOR $2 AT $3 PER YARD

Figure 4-2.—Measurement fractions.

PARTITIVE FRACTIONS

The difference between measurement fractions and partitive fractions is explained as follows: Measurement fractions result when we determine how many pieces of a given size can be cut from a larger piece. Partitive fractions result when we cut a number of pieces of equal size from a larger piece and then determine the size of each smaller piece. For example, if 4 equal lengths of pipe are to be cut from a 3-foot pipe, what is the size of each piece? If the problem had read that 3 equal lengths were to be cut from a 6-foot pipe, we could find the size of each pipe by dividing the number of equal lengths into the overall length. Thus, since 6/3 is 2, each piece would be 2 feet long. By this same reasoning in the example, we divide the overall length by the number of equal parts to get the size of the individual pieces; that is, 3/4 foot. The partitioned 6-foot and 3-foot pipes are shown in figure 4-3.

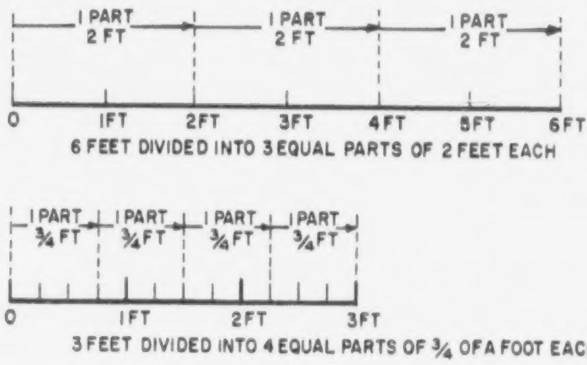


Figure 4-3.—Partitive fractions

EXPRESSING RELATIONSHIPS

When a fraction is used to express a relationship, the numerator and denominator take on individual significance. In this frame of reference, 3/4 means 3 out of 4, or 3 parts in 4, or the ratio of 3 to 4. For example, if 1 out of 3 of the men in a division are on liberty, then it would be correct to state that 1/3 of the division are on liberty. Observe that neither of these ways of expressing the relationship tells us the actual number of men; the relationship itself is the important thing.

Practice problems.

1. What fraction of 1 foot is 11 inches?
2. Represent 3 out of 8 as a fraction.
3. Write the fractions that indicate the relationship of 2 to 3; 8 divided by 9; and 6 out of 7 equal parts.
4. The number $6\frac{3}{5}$ means $6 \underline{\ \ ?\ \ } \frac{3}{5}$

Answers:

1. 11/12
2. 3/8
3. 2/3; 8/9; 6/7
4. plus

EQUIVALENT FRACTIONS

It will be recalled that any number divided by itself is 1. For example, 1/1, 2/2, 3/3, 4/4, and all other numbers formed in this way, have the value 1. Furthermore, any number multiplied by 1 is equivalent to the number itself. For example, 1 times 2 is 2, 1 times 3 is 3, 1 times 1/2 is 1/2, etc.

These facts are used in changing the form of a fraction to an equivalent form which is more convenient for use in a particular problem.

For example, if 1 in the form $\frac{2}{2}$ is multiplied by $\frac{3}{5}$, the product will still have a value of $\frac{3}{5}$ but will be in a different form, as follows:

$$\frac{2}{2} \cdot \frac{3}{5} = \frac{2 \cdot 3}{2 \cdot 5} = \frac{6}{10}$$

Figure 4-4 shows that $\frac{3}{5}$ of line a is equal to $\frac{6}{10}$ of line b where line a equals line b. Line a is marked off in fifths and line b is marked off in tenths. It can readily be seen that $\frac{6}{10}$ and $\frac{3}{5}$ measure distances of equal length.

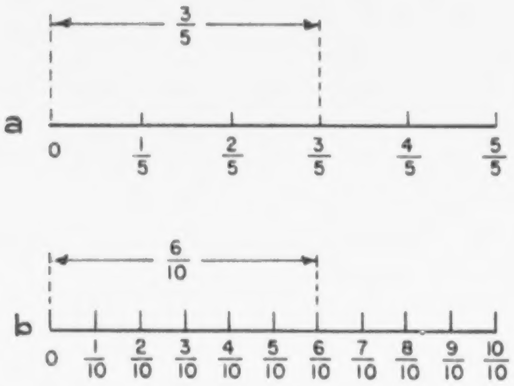


Figure 4-4.—Equivalent fractions.

The markings on a ruler show equivalent fractions. The major division of an inch divides it into two equal parts. One of these parts represents $\frac{1}{2}$. The next smaller markings divide the inch into four equal parts. It will be noted that two of these parts represent the same distance as $\frac{1}{2}$; that is, $\frac{2}{4}$ equals $\frac{1}{2}$. Also, the next smaller markings break the inch into 8 equal parts. How many of these parts are equivalent to $\frac{1}{2}$ inch? The answer is found by noting that $\frac{4}{8}$ equals $\frac{1}{2}$.

Practice problems. Using the divisions on a ruler for reference, complete the following exercise:

1. $\frac{1}{4} = \frac{?}{8}$
2. $\frac{1}{8} = \frac{?}{16}$
3. $\frac{3}{4} = \frac{?}{16}$
4. $\frac{1}{4} = \frac{?}{16}$

Answers:

1. 2
2. 2
3. 12
4. 4

A review of the foregoing exercise will reveal that in each case the right-hand fraction could be formed by multiplying both the numerator and the denominator of the left-hand fraction by the same number. In each case the number may be determined by dividing the denominator of the right-hand fraction by the denominator of the left-hand fraction. Thus in problem 1, both terms of $\frac{1}{4}$ were multiplied by 2. In problem 3, both terms were multiplied by 4. It is seen that multiplying both terms of a fraction by the same number does not change the value of the fraction.

Since $\frac{1}{2}$ equals $\frac{2}{4}$, the reverse must also be true; that is $\frac{2}{4}$ must be equal to $\frac{1}{2}$. This can likewise be verified on a ruler. We have already seen that $\frac{4}{8}$ is the same as $\frac{1}{2}$, $\frac{12}{16}$ equals $\frac{3}{4}$, and $\frac{2}{8}$ equals $\frac{1}{4}$. We see that dividing both terms of a fraction by the same number does not change the value of the fraction.

FUNDAMENTAL RULE OF FRACTIONS

The foregoing results are combined to form the fundamental rule of fractions, which is stated as follows: Multiplying or dividing both terms of a fraction by the same number does not change the value of the fraction. This is one of the most important rules used in dealing with fractions.

The following examples show how the fundamental rule is used:

1. Change 1/4 to twelfths. This problem is set up as follows:

$$\frac{1}{4} = \frac{?}{12}$$

The first step is to determine how many 4's are contained in 12. The answer is 3, so we know that the multiplier for both terms of the fraction is 3, as follows:

$$\frac{3}{3} \cdot \frac{1}{4} = \frac{3}{12}$$

2. What fraction with a numerator of 6 is equal to 3/4?

SOLUTION: $\frac{6}{?} = \frac{3}{4}$

We note that 6 contains 3 twice; therefore we need to double the numerator of the right-hand fraction to make it equivalent to the numerator of the fraction we seek. We multiply both terms of 3/4 by 2, obtaining 8 as the denominator of the new fraction, as follows:

$$\frac{6}{8} = \frac{3}{4} \cdot \frac{2}{2}$$

3. Change 6/16 to eighths.

SOLUTION: $\frac{6}{16} = \frac{?}{8}$

We note that the denominator of the fraction which we seek is 1/2 as large as the denominator of the original fraction. Therefore the new fraction may be formed by dividing both terms of the original fraction by 2, as follows:

$$\frac{6 \div 2}{16 \div 2} = \frac{3}{8}$$

Practice problems. Supply the missing number in each of the following:

1. $\frac{3}{8} = \frac{30}{?}$
2. $\frac{44}{48} = \frac{?}{12}$
3. $\frac{?}{90} = \frac{3}{10}$
4. $\frac{1}{6} = \frac{6}{?}$
5. $\frac{1}{?} = \frac{12}{72}$
6. $\frac{3}{5} = \frac{?}{25}$

Answers:

1. 80
2. 11
3. 27
4. 36
5. 6
6. 15

REDUCTION TO LOWEST TERMS

It is frequently desirable to change a fraction to an equivalent fraction with the smallest possible terms; that is, with the smallest possible numerator and denominator. This process is called REDUCTION. Thus, $\frac{6}{30}$ reduced to lowest terms is $\frac{1}{5}$. Reduction can be accomplished by finding the largest factor that is common to both the numerator and denominator and dividing both of these terms by it. Dividing both terms of the preceding example by 6 reduces the fraction to lowest terms. In computation, fractions should usually be reduced to lowest terms where possible.

If the greatest common factor cannot readily be found, any common factor may be removed and the process repeated until the fraction is in lowest terms: Thus, $\frac{18}{48}$ could first be divided by 2 and then by 3.

$$\frac{18 \div 2}{48 \div 2} = \frac{9}{24}$$

$$\frac{9 \div 3}{24 \div 3} = \frac{3}{8}$$

Practice problems. Reduce the following fractions to lowest terms:

1. $\frac{18}{48}$
2. $\frac{15}{20}$
3. $\frac{35}{56}$
4. $\frac{12}{60}$
5. $\frac{18}{24}$
6. $\frac{9}{144}$

Answers:

1. $\frac{3}{8}$
2. $\frac{3}{4}$
3. $\frac{5}{8}$
4. $\frac{1}{5}$
5. $\frac{3}{4}$
6. $\frac{1}{16}$

IMPROPER FRACTIONS

Although the "improper" fraction is really quite "proper" mathematically, it is usually customary to change it to a mixed number. A recipe may call for $1\frac{1}{2}$ cups of milk, but would not call for $\frac{3}{2}$ cups of milk.

Since a fraction is an indicated division, a method is already known for reduction of improper fractions to mixed numbers. The improper fraction $\frac{8}{3}$ may be considered as the division of 8 by 3. This division is carried out as follows:

$$\begin{array}{l} \quad 2 \text{ R } 2 = 2\frac{2}{3} \\ 3\overline{)8} \\ \quad \underline{6} \\ \quad 2 \end{array}$$

The truth of this can be verified another way: If 1 equals $\frac{3}{3}$, then 2 equals $\frac{6}{3}$. Thus,

$$2\frac{2}{3} = 2 + \frac{2}{3} = \frac{6}{3} + \frac{2}{3} = \frac{8}{3}$$

These examples lead to the following conclusion, which is stated as a rule: To change an improper fraction to a mixed number, divide the numerator by the denominator and write the fractional part of the quotient in lowest terms.

Practice problems. Change the following fractions to mixed numbers:

1. 31/20
2. 33/9
3. 65/20
4. 45/8

Answers:

1. $1\frac{11}{20}$
2. $3\frac{2}{3}$
3. $3\frac{1}{4}$
4. $5\frac{5}{8}$

OPERATING WITH MIXED NUMBERS

In computation, mixed numbers are often unwieldy. As it is possible to change any improper fraction to a mixed number, it is likewise possible to change any mixed number to an improper fraction. The problem can be reduced to the finding of an equivalent fraction and a simple addition.

EXAMPLE: Change $2\frac{1}{5}$ to an improper fraction.

SOLUTION:

Step 1: Write $2\frac{1}{5}$ as a whole number plus a fraction, $2 + \frac{1}{5}$.

Step 2: Change 2 to an equivalent fraction with a denominator of 5, as follows:

$$\frac{2}{1} = \frac{?}{5}$$

$$\frac{2(5)}{1(5)} = \frac{10}{5}$$

Step 3: Add $\frac{10}{5} + \frac{1}{5} = \frac{11}{5}$

Thus, $2\frac{1}{5} = \frac{11}{5}$

EXAMPLE: Write $5\frac{2}{9}$ as an improper fraction.

SOLUTION: $5\frac{2}{9} = 5 + \frac{2}{9}$

$$\frac{5}{1} = \frac{?}{9}$$

$$\frac{5(9)}{1(9)} = \frac{45}{9}$$

$$\frac{45}{9} + \frac{2}{9} = \frac{47}{9}$$

Thus, $5\frac{2}{9} = \frac{47}{9}$

In each of these examples, notice that the multiplier used in step 2 is the same number as the denominator of the fractional part of the original mixed number. This leads to the following conclusion, which is stated as a rule: To change a mixed number to an improper fraction, multiply the whole-number part by the denominator of the fractional part and add the numerator to this product. The result is the numerator of the improper fraction; its denominator is the same as the denominator of the fractional part of the original mixed number.

Practice problems. Change the following mixed numbers to improper fractions:

1. $1\frac{1}{5}$
2. $2\frac{11}{20}$
3. $3\frac{2}{7}$
4. $4\frac{3}{10}$

Answers:

1. $\frac{6}{5}$
2. $\frac{51}{20}$
3. $\frac{23}{7}$
4. $\frac{43}{10}$

NEGATIVE FRACTIONS

A fraction preceded by a minus sign is negative. Any negative fraction is equivalent to a positive fraction multiplied by -1. For example,

$$-\frac{2}{5} = -1\left(\frac{2}{5}\right)$$

The number $-\frac{2}{5}$ is read "minus two-fifths."

We know that the quotient of two numbers with unlike signs is negative. Therefore,

$$\frac{-2}{5} = -\frac{2}{5} \text{ and } \frac{2}{-5} = -\frac{2}{5}$$

This indicates that a negative fraction is equivalent to a fraction with either a negative numerator or a negative denominator.

The fraction $\frac{2}{-5}$ is read "two over minus five." The fraction $\frac{-2}{5}$ is read "minus two over five."

A minus sign in a fraction can be moved about at will. It can be placed before the numerator, before the denominator, or before the fraction itself. Thus,

$$\frac{-2}{5} = \frac{2}{-5} = -\frac{2}{5}$$

Moving the minus sign from numerator to denominator, or vice versa, is equivalent to multiplying the terms of the fraction by -1. This is shown in the following examples:

$$\frac{-2(-1)}{5(-1)} = \frac{2}{-5} \text{ and } \frac{2(-1)}{-5(-1)} = \frac{-2}{5}$$

A fraction may be regarded as having three signs associated with it—the sign of the numerator, the sign of the denominator, and the sign preceding the fraction. Any two of these signs may be changed without changing the value of the fraction. Thus,

$$-\frac{3}{4} = \frac{-3}{4} = \frac{3}{-4} = -\frac{-3}{-4}$$

OPERATIONS WITH FRACTIONS

It will be recalled from the discussion of denominate numbers that numbers must be of the same denomination to be added. We can add pounds to pounds, pints to pints, but not ounces to pints. If we think of fractions loosely as denominate numbers, it will be seen that the rule of likeness applies also to fractions. We can add eighths to eighths, fourths to fourths, but not eighths to fourths. To add $\frac{1}{5}$ inch to $\frac{2}{5}$ inch we simply add the numerators and retain the denominator unchanged. The denomination is fifths; as with denominate numbers, we add 1 fifth to 2 fifths to get 3 fifths, or $\frac{3}{5}$.

LIKE AND UNLIKE FRACTIONS

We have shown that like fractions are added by simply adding the numerators and keeping the denominator. Thus,

$$\frac{3}{8} + \frac{2}{8} = \frac{3+2}{8} = \frac{5}{8}$$

or

$$\frac{5}{16} + \frac{2}{16} = \frac{7}{16}$$

Similarly we can subtract like fractions by subtracting the numerators.

$$\frac{7}{8} - \frac{2}{8} = \frac{7-2}{8} = \frac{5}{8}$$

The following examples will show that like fractions may be divided by dividing the numerator of the dividend by the numerator of the divisor.

$$\frac{3}{8} \div \frac{1}{8} = ?$$

SOLUTION: We may state the problem as a question: "How many times does $\frac{1}{8}$ appear in $\frac{3}{8}$, or how many times may $\frac{1}{8}$ be taken from $\frac{3}{8}$?"

$$3/8 - 1/8 = 2/8 \quad (1)$$

$$2/8 - 1/8 = 1/8 \quad (2)$$

$$1/8 - 1/8 = 0/8 = 0 \quad (3)$$

We see that 1/8 can be subtracted from 3/8 three times. Therefore,

$$3/8 \div 1/8 = 3$$

When the denominators of fractions are unequal, the fractions are said to be unlike. Addition, subtraction, or division cannot be performed directly on unlike fractions. The proper application of the fundamental rule, however, can change their form so that they become like fractions; then all the rules for like fractions apply.

LOWEST COMMON DENOMINATOR

To change unlike fractions to like fractions, it is necessary to find a COMMON DENOMINATOR and it is usually advantageous to find the LOWEST COMMON DENOMINATOR (LCD). This is nothing more than the least common multiple of the denominators.

Least Common Multiple

If a number is a multiple of two or more different numbers, it is called a COMMON MULTIPLE. Thus, 24 is a common multiple of 6 and 2. There are many common multiples of these numbers. The numbers 36, 48, and 54, to name a few, are also common multiples of 6 and 2.

The smallest of the common multiples of a set of numbers is called the LEAST COMMON MULTIPLE. It is abbreviated LCM. The least common multiple of 6 and 2 is 6. To find the least common multiple of a set of numbers, first separate each of the numbers into prime factors.

Suppose that we wish to find the LCM of 14, 24, and 30. Separating these numbers into prime factors we have

$$\begin{aligned} 14 &= 2 \cdot 7 \\ 24 &= 2^3 \cdot 3 \\ 30 &= 2 \cdot 3 \cdot 5 \end{aligned}$$

The LCM will contain each of the various prime factors shown. Each prime factor is used the greatest number of times that it occurs in any one of the numbers. Notice that 3, 5, and 7 each occur only once in any one number. On the other hand, 2 occurs three times in one number. We get the following result:

$$\begin{aligned} \text{LCM} &= 2^3 \cdot 3 \cdot 5 \cdot 7 \\ &= 840 \end{aligned}$$

Thus, 840 is the least common multiple of 14, 24, and 30.

Greatest Common Divisor

The largest number that can be divided into each of two or more given numbers without a remainder is called the GREATEST COMMON DIVISOR of the given numbers. It is abbreviated GCD. It is also sometimes called the HIGHEST COMMON FACTOR.

In finding the GCD of a set of numbers, separate the numbers into prime factors just as for LCM. The GCD is the product of only those factors that appear in all of the numbers. Notice in the example of the previous section that 2 is the greatest common divisor of 14, 24, and 30.

Find the GCD of 650, 900, and 700. The procedure is as follows:

$$\begin{aligned} 650 &= 2 \cdot 5^2 \cdot 13 \\ 900 &= 2^2 \cdot 3^2 \cdot 5^2 \\ 700 &= 2^2 \cdot 5^2 \cdot 7 \\ \text{GCD} &= 2 \cdot 5^2 = 50 \end{aligned}$$

Notice that 2 and 5^2 are factors of each number. The greatest common divisor is 2 x 25 = 50.

USING THE LCD

Consider the example

$$\frac{1}{2} + \frac{1}{3}$$

The numbers 2 and 3 are both prime; so the LCD is 6.

Therefore $\frac{1}{2} = \frac{3}{6}$

and $\frac{1}{3} = \frac{2}{6}$

Thus, the addition of $\frac{1}{2}$ and $\frac{1}{3}$ is performed as follows:

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

In the example

$$\frac{1}{5} + \frac{3}{10}$$

10 is the LCD.

Therefore,

$$\begin{aligned}\frac{1}{5} + \frac{3}{10} &= \frac{2}{10} + \frac{3}{10}\\ &= \frac{5}{10} = \frac{1}{2}\end{aligned}$$

Practice problems. Change the fractions in each of the following groups to like fractions with least common denominators:

1. $\frac{1}{3}, \frac{1}{6}$
2. $\frac{5}{12}, \frac{2}{3}$
3. $\frac{1}{2}, \frac{1}{4}, \frac{2}{3}$
4. $\frac{1}{6}, \frac{3}{10}, \frac{1}{5}$

Answers:

1. $\frac{2}{6}, \frac{1}{6}$
2. $\frac{5}{12}, \frac{8}{12}$
3. $\frac{6}{12}, \frac{3}{12}, \frac{8}{12}$
4. $\frac{5}{30}, \frac{9}{30}, \frac{6}{30}$

ADDITION

It has been shown that in adding like fractions we add the numerators. In adding unlike fractions, the fractions must first be changed so that they have common denominators. We apply these same rules in adding mixed numbers. It will be remembered that a mixed number is an indicated sum. Thus, $2\frac{1}{3}$ is really $2 + \frac{1}{3}$. Adding can be done in any order. The following examples will show the application of these rules:

EXAMPLE:

$$\begin{array}{r} 2\frac{1}{3} \\ 3\frac{1}{3} \\ \hline 5\frac{2}{3} \end{array}$$

This could have been written as follows:

$$\begin{array}{l} 2 + \frac{1}{3} \\ 3 + \frac{1}{3} \\ \hline 5 + \frac{2}{3} = 5\frac{2}{3} \end{array}$$

EXAMPLE:

$$\begin{array}{r} 4\frac{5}{7} \\ 6\frac{3}{7} \\ \hline 10\frac{8}{7} \end{array}$$

Here we change $\frac{8}{7}$ to the mixed number $1\frac{1}{7}$. Then

$$\begin{aligned}10\frac{8}{7} &= 10 + 1 + \frac{1}{7}\\ &= 11\frac{1}{7}\end{aligned}$$

EXAMPLE:

Add

$$\begin{array}{r} \frac{1}{4} \\ 2\frac{2}{3} \\ \hline \end{array}$$

We first change the fractions so that they are like and have the least common denominator and then proceed as before.

$$\begin{array}{rcl} \frac{1}{4} &=& \frac{3}{12} \\ 2\frac{2}{3} &=& 2\frac{8}{12} \\ \hline && 2\frac{11}{12} \end{array}$$

EXAMPLE:

Add

$$\begin{array}{rcl} 4\frac{5}{8} &=& 4\frac{5}{8} \\ 2\frac{1}{2} &=& 2\frac{4}{8} \\ \frac{1}{4} &=& \frac{2}{8} \\ \hline && 6\frac{11}{8} \end{array}$$

Since $\frac{11}{8}$ equals $1\frac{3}{8}$, the final answer is found as follows:

$$6\,\frac{11}{8} = 6 + 1 + \frac{3}{8}$$

$$= 7\,\frac{3}{8}$$

Practice problems. Add, and reduce the sums to simplest terms:

1. $1\frac{1}{7}$	2. $\frac{3}{4}$	3. $6\frac{2}{5}$	4. $\frac{5}{8}$	5. $4\frac{1}{2}$
$\underline{2\frac{3}{4}}$	$\underline{1\frac{1}{2}}$	$\underline{3\frac{1}{4}}$	$\underline{2\frac{3}{20}}$	$\underline{1\frac{1}{8}}$

Answers:

1. $3\frac{25}{28}$ 2. $2\frac{1}{4}$ 3. $9\frac{13}{20}$ 4. $2\frac{31}{40}$ 5. $5\frac{5}{8}$

The following example demonstrates a practical application of addition of fractions:

EXAMPLE: Find the total length of the piece of metal shown in figure 4-5 (A).

SOLUTION: First indicate the sum as follows:

$$\frac{9}{16} + \frac{3}{4} + \frac{7}{8} + \frac{3}{4} + \frac{9}{16} = ?$$

Changing to like fractions and adding numerators,

$$\frac{9}{16} + \frac{12}{16} + \frac{14}{16} + \frac{12}{16} + \frac{9}{16} = \frac{56}{16}$$

$$= 3\,\frac{8}{16}$$

$$= 3\,\frac{1}{2}$$

The total length is $3\frac{1}{2}$ inches.

Practice problem. Find the distance from the center of the first hole to the center of the last hole in the metal plate shown in figure 4-5 (B).

Answer: $2\frac{7}{16}$ inches

SUBTRACTION

The rule of likeness applies in the subtraction of fractions as well as in addition. Some examples will show that cases likely to arise may be solved by use of ideas previously developed.

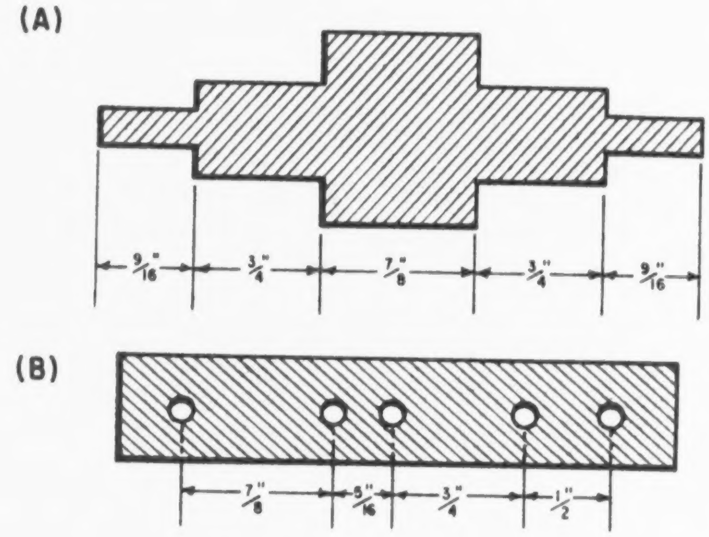


Figure 4-5.—Adding fractions to obtain total length or spacing.

EXAMPLE: Subtract $1\frac{1}{3}$ from $5\frac{2}{3}$

$$\begin{array}{r} 5\frac{2}{3} \\ \underline{1\frac{1}{3}} \\ 4\frac{1}{3} \end{array}$$

We see that whole numbers are subtracted from whole numbers; fractions from fractions.

EXAMPLE: Subtract $\frac{1}{8}$ from $\frac{4}{5}$

$$\begin{array}{r} \frac{4}{5} \\ \underline{\frac{1}{8}} \end{array}$$

Changing to like fractions with an LCD, we have

$$\begin{array}{r} \frac{32}{40} \\ \underline{\frac{5}{40}} \\ \frac{27}{40} \end{array}$$

EXAMPLE: Subtract $\frac{11}{12}$ from $3\frac{2}{3}$

$$3\frac{2}{3} = 3\frac{8}{12}$$
$$\frac{11}{12} = \frac{11}{12}$$

Regrouping $3\frac{8}{12}$ we have

$$2 + 1 + \frac{8}{12} = 2 + \frac{12}{12} + \frac{8}{12}$$

Then

$$3\frac{2}{3} = 2\frac{20}{12}$$
$$\frac{11}{12} = \frac{11}{12}$$
$$2\frac{9}{12} = 2\frac{3}{4}$$

Practice problems. Subtract the lower number from the upper number and reduce the difference to simplest terms:

1. $\frac{7}{9}$	2. $\frac{2}{3}$	3. $5\frac{5}{12}$	4. 5	5. $2\frac{3}{8}$
$\frac{1}{6}$	$\frac{1}{3}$	$2\frac{7}{12}$	$2\frac{2}{3}$	$\frac{5}{8}$

Answers:

1. $\frac{11}{18}$ 2. $\frac{1}{3}$ 3. $2\frac{5}{6}$ 4. $2\frac{1}{3}$ 5. $1\frac{3}{4}$

The following problem demonstrates subtraction of fractions in a practical situation.

EXAMPLE: What is the length of the dimension marked X on the machine bolt shown in figure 4-6 (A)?

SOLUTION: Total the lengths of the known parts.

$$\frac{1}{4} + \frac{1}{64} + \frac{1}{2} = \frac{16}{64} + \frac{1}{64} + \frac{32}{64} = \frac{49}{64}$$

Subtract this sum from the overall length.

$$2 = 1\frac{64}{64}$$
$$\frac{49}{64} = \frac{49}{64}$$
$$1\frac{15}{64}$$

The answer is $1\frac{15}{64}$ inch.

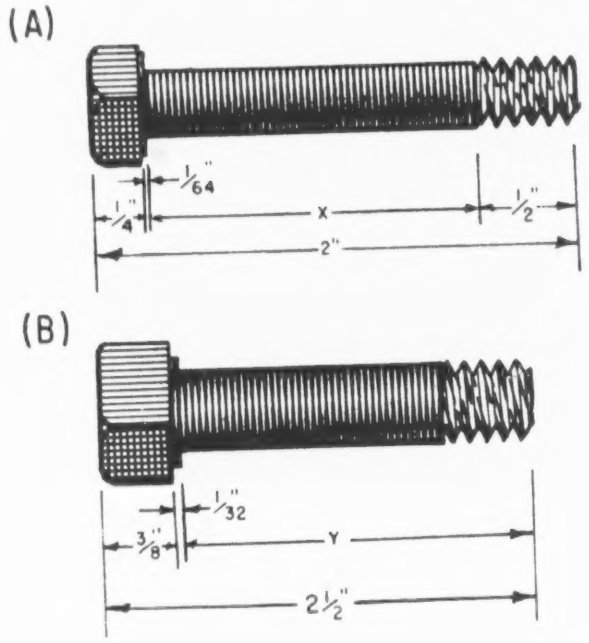


Figure 4-6.—Finding unknown dimensions by subtracting fractions.

Practice problem. Find the length of the dimension marked Y on the machine bolt in figure 4-6 (B).

MULTIPLICATION

The fact that multiplication by a fraction does not increase the value of the product may confuse those who remember the definition of multiplication presented earlier for whole numbers. It was stated that 4(5) means 5 is taken as an addend 4 times. How is it then that $\frac{1}{2}(4)$ is 2, a number less than 4? Obviously our idea of multiplication must be broadened.

Consider the following products:

$$4(4) = 16$$
$$3(4) = 12$$
$$2(4) = 8$$
$$1(4) = 4$$

$$\frac{1}{2}(4) = 2$$

$$\frac{1}{4}(4) = 1$$

Notice that as the multiplier decreases, the product decreases, until, when the multiplier is a fraction, the product is less than 4 and continues to decrease as the fraction decreases. The fraction introduces the "part of" idea: $\frac{1}{2}(4)$ means $\frac{1}{2}$ of 4; $\frac{1}{4}(4)$ means $\frac{1}{4}$ of 4.

The definition of multiplication stated for whole numbers may be extended to include fractions. Since 4(5) means that 5 is to be used 4 times as an addend, we can say that with fractions the numerator of the multiplier tells how many times the numerator of the multiplicand is to be used as an addend. By the same reasoning, the denominator of the multiplier tells how many times the denominator of the multiplicand is to be used as an addend. The following examples illustrate the use of this idea:

1. The fraction $\frac{1}{12}$ is multiplied by the whole number 4 as follows:

$$\begin{aligned} 4 \times \frac{1}{12} &= \frac{4}{1} \times \frac{1}{12} \\ &= \frac{1 + 1 + 1 + 1}{12} \\ &= \frac{4}{12} = \frac{1}{3} \end{aligned}$$

This example shows that 4 (1/12) is the same as $\frac{4(1)}{12}$.

Another way of thinking about the multiplication of 1/12 by 4 is as follows:

$$\begin{aligned} 4 \times \frac{1}{12} &= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} \\ &= \frac{4}{12} = \frac{1}{3} \end{aligned}$$

2. The fraction 2/3 is multiplied by 1/2 as follows:

$$\begin{aligned} \frac{1}{2} \times \frac{2}{3} &= \frac{2}{6} \\ &= \frac{1}{3} \end{aligned}$$

From these examples a general rule is developed: To find the product of two or more fractions multiply their numerators together and write the result as the numerator of the product; multiply their denominators and write the result as the denominator of the product; reduce the answer to lowest terms.

In using this rule with whole numbers, write each whole number as a fraction with 1 as the denominator. For example, multiply 4 times 1/12 as follows:

$$\begin{aligned} 4 \times \frac{1}{12} &= \frac{4}{1} \times \frac{1}{12} \\ &= \frac{4}{12} = \frac{1}{3} \end{aligned}$$

In using this rule with mixed numbers, rewrite all mixed numbers as improper fractions before applying the rule, as follows:

$$\begin{aligned} 2\frac{1}{3} \times \frac{1}{2} &= \frac{7}{3} \times \frac{1}{2} \\ &= \frac{7}{6} \end{aligned}$$

A second method of multiplying mixed numbers makes use of the distributive law. This law states that a multiplier applied to a two-part expression is distributed over both parts. For example, to multiply $6\frac{1}{3}$ by 4 we may rewrite $6\frac{1}{3}$ as 6 + 1/3. Then the problem can be written as 4(6 + 1/3) and the multiplication proceeds as follows:

$$\begin{aligned} 4(6 + 1/3) &= 24 + 4/3 \\ &= 25 + 1/3 \\ &= 25\frac{1}{3} \end{aligned}$$

Cancellation

Computation can be considerably reduced by dividing out (CANCELLING) factors common to both the numerator and the denominator. We recognize a fraction as an indicated division. Thinking of $\frac{6}{9}$ as an indicated division, we remember that we can simplify division by showing both dividend and divisor as the indicated products of their factors and then dividing like factors, or canceling. Thus,

$$\frac{6}{9} = \frac{2 \times 3}{3 \times 3}$$

Dividing the factor 3 in the numerator by 3 in the denominator gives the following simplified result:

$$\frac{2 \times \overset{1}{\cancel{3}}}{3 \times \underset{1}{\cancel{3}}} = \frac{2}{3}$$

This method is most advantageous when done before any other computation. Consider the example,

$$\frac{1}{3} \times \frac{3}{2} \times \frac{2}{5}$$

The product in factored form is

$$\frac{1 \times 3 \times 2}{3 \times 2 \times 5}$$

Rather than doing the multiplying and then reducing the result $\frac{6}{30}$, it is simpler to cancel like factors first, as follows:

$$\frac{1 \times \overset{1}{\cancel{3}} \times \overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}} \times \underset{1}{\cancel{2}} \times 5} = \frac{1}{5}$$

Likewise,

$$\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{\overset{1}{\cancel{2}}}{\cancel{6}}}{\underset{\underset{1}{\cancel{2}}}{\cancel{4}}} \times \frac{5}{9} = \frac{5}{9}$$

Here we mentally factor 6 to the form 3 x 2, and 4 to the form 2 x 2. Cancellation is a valuable tool in shortening operations with fractions.

The general rule may be applied to mixed numbers by simply changing them to improper fractions.

Thus,

$$2\frac{1}{4} \times 3\frac{1}{3} = ?$$

$$\frac{9}{4} \times \frac{10}{3} = \frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{4}}} \times \frac{\overset{5}{\cancel{10}}}{\underset{1}{\cancel{3}}} = \frac{15}{2}$$

$$= 7\frac{1}{2}$$

Practice problems. Determine the following products, using the general rule and canceling where possible:

1. $\frac{5}{8} \times 12$
2. $\frac{1}{2} \times \frac{1}{3} \times \frac{2}{5}$
3. $5 \times \frac{4}{9}$
4. $\frac{3}{4} \times 6$
5. $\frac{1}{3} \times \frac{2}{3}$
6. $\frac{4}{3} \times \frac{1}{6}$

Answers:

1. $7\frac{1}{2}$
2. $\frac{1}{15}$
3. $2\frac{2}{9}$
4. $4\frac{1}{2}$
5. $\frac{2}{9}$
6. $\frac{2}{9}$

The following problem illustrates the multiplication of fractions in a practical situation.

EXAMPLE: Find the distance between the center lines of the first and fifth rivets connecting the two metal plates shown in figure 4-7 (A).

SOLUTION: The distance between two adjacent rivets, centerline to centerline, is 4 1/2 times the diameter of one of them.

Thus,

$$\begin{aligned} 1 \text{ space} &= 4\frac{1}{2} \times \frac{5}{8} \\ &= \frac{9}{2} \times \frac{5}{8} \\ &= \frac{45}{16} \end{aligned}$$

There are 4 such spaces between the first and fifth rivets. Therefore, the total distance, D, is found as follows:

$$D = \overset{1}{\cancel{4}} \times \frac{45}{\underset{4}{\cancel{16}}} = \frac{45}{4} = 11\frac{1}{4}$$

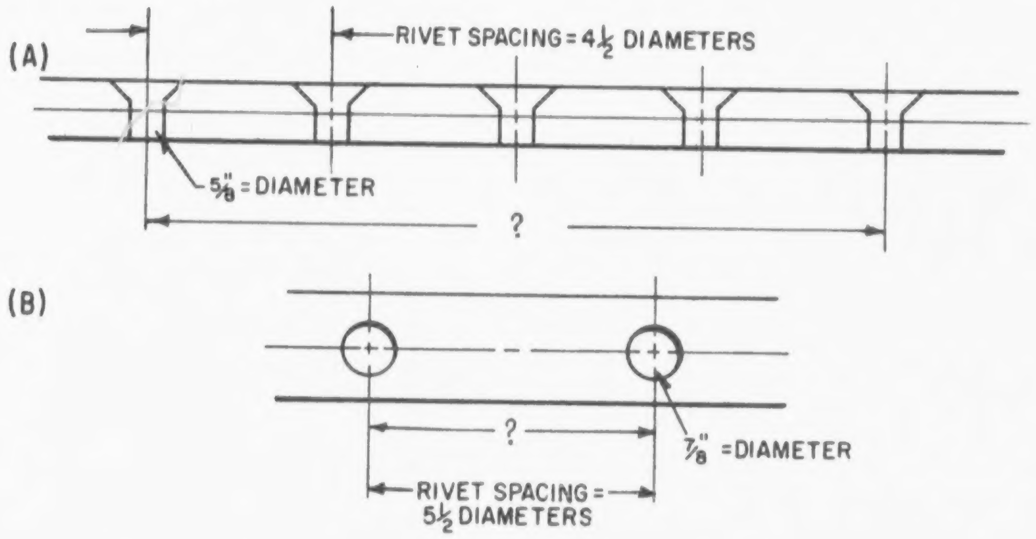

(A)
RIVET SPACING = $4\frac{1}{2}$ DIAMETERS
$\frac{5}{8}$" = DIAMETER
?
(B)
?
$\frac{7}{8}$" = DIAMETER
RIVET SPACING = $5\frac{1}{2}$ DIAMETERS

Figure 4-7.—Application of multiplication of fractions in determining rivet spacing.

The distance is $11\frac{1}{4}$ inches

Practice problem. Find the distance between the centers of the two rivets shown in figure 4-7 (B).

Answer: $4\frac{13}{16}$ inches

DIVISION

There are two methods commonly used for performing division with fractions. One is the common denominator method and the other is the reciprocal method.

Common Denominator Method

The common denominator method is an adaptation of the method of like fractions. The rule is as follows: Change the dividend and divisor to like fractions and divide the numerator of the dividend by the numerator of the divisor. This method can be demonstrated with whole numbers, first changing them to fractions with 1 as the denominator. For example, 12 ÷ 4 can be written as follows:

$$12 \div 4 = \frac{12}{1} \div \frac{4}{1}$$

$$= \frac{12 \div 4}{1 \div 1}$$

$$= \frac{12 \div 4}{1}$$

$$= 3$$

If the dividend and divisor are both fractions, as in 1/3 divided by 1/4, we proceed as follows:

$$\frac{1}{3} \div \frac{1}{4} = \frac{4}{12} \div \frac{3}{12}$$

$$= \frac{4 \div 3}{12 \div 12}$$

$$= \frac{4 \div 3}{1}$$

$$= 4 \div 3 = 1\frac{1}{3}$$

Reciprocal Method

The word "reciprocal" denotes an interchangeable relationship. It is used in mathematics to describe a specific relationship between two numbers. We say that two numbers are reciprocals of each other if their product is one. In the example $4 \times \frac{1}{4} = 1$, the fractions $\frac{4}{1}$ and $\frac{1}{4}$ are reciprocals. Notice the interchangeability: 4 is the reciprocal of $\frac{1}{4}$ and $\frac{1}{4}$ is the reciprocal of 4.

What is the reciprocal of $\frac{3}{7}$? It must be a number which, when multiplied by $\frac{3}{7}$, produces the product, 1. Therefore,

$$\frac{3}{7} \times ? = 1$$

$$\overset{1}{\underset{1}{\frac{\cancel{3}}{\cancel{7}}}} \times \overset{1}{\underset{1}{\frac{\cancel{7}}{\cancel{3}}}} = 1$$

We see that $\frac{7}{3}$ is the only number that could fulfill the requirement. Notice that the numerator and denominator of $\frac{3}{7}$ were simply interchanged to get its reciprocal. If we know a number, we can always find its reciprocal by dividing 1 by the number. Notice this principle in the following examples:

1. What is the reciprocal of 7?

$$1 \div 7 = \frac{1}{7}$$

Check:

$$\frac{\cancel{7}}{\cancel{1}} \times \frac{\cancel{1}}{\cancel{7}} = 1$$

Notice that the cancellation process in this example does not show the usual 1's which result when dividing a number into itself. For example, when 7 cancels 7, the quotient 1 could be shown beside each of the 7's. However, since 1 as a factor has the same effect whether it is written in or simply understood, the 1's need not be written.

2. What is the reciprocal of $\frac{3}{8}$?

$$1 \div \frac{3}{8} = \frac{8}{8} \div \frac{3}{8}$$
$$= 8 \div 3, \text{ or } \frac{8}{3}$$

Check:

$$\frac{\cancel{3}}{\cancel{8}} \times \frac{\cancel{8}}{\cancel{3}} = 1.$$

3. What is the reciprocal of $\frac{5}{2}$?

SOLUTION:

$$1 \div \frac{5}{2} = \frac{2}{2} \div \frac{5}{2}$$
$$= 2 \div 5$$
$$= \frac{2}{5}$$

Check:

$$\frac{\cancel{5}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{5}} = 1$$

4. What is the reciprocal of $3\frac{1}{8}$?

SOLUTION:

$$1 \div 3\frac{1}{8} = \frac{8}{8} \div \frac{25}{8}$$
$$= 8 \div 25$$
$$= \frac{8}{25}$$

Check:

$$\frac{\cancel{25}}{\cancel{8}} \times \frac{\cancel{8}}{\cancel{25}} = 1$$

The foregoing examples lead to the rule for finding the reciprocal of any number: The reciprocal of a number is the fraction formed when 1 is divided by the number. (If the final result is a whole number, it can be considered as a fraction whose denominator is 1.) A shortcut rule which is purely mechanical and does not involve reasoning may be stated as follows: To find the reciprocal of a number, express the number as a fraction and then invert the fraction.

When the numerator of a fraction is 1, the reciprocal is a whole number. The smaller the fraction, the greater is the reciprocal. For example, the reciprocal of $\frac{1}{1,000}$ is 1,000. Also, the reciprocal of any whole number is a proper fraction. Thus the reciprocal of 50 is $\frac{1}{50}$.

Practice problems. Write the reciprocal of each of the following numbers:

1. 4 2. $\frac{1}{3}$ 3. $2\frac{1}{2}$ 4. 17 5. $\frac{3}{2}$ 6. $\frac{5}{1}$

Answers:

1. $\frac{1}{4}$ 2. 3 3. $\frac{2}{5}$ 4. $\frac{1}{17}$ 5. $\frac{2}{3}$ 6. $\frac{1}{5}$

The reciprocal method of division makes use of the close association of multiplication and division. In any division problem, we must find the answer to the following question: What number multiplied by the divisor yields the dividend? For example, if the problem is to divide 24 by 6, we must find the factor which, when multiplied by 6, yields 24. Experience tells us that the number we seek is 1/6 of 24. Thus, we may rewrite the problem as follows:

$$24 \div 6 = \frac{1}{6} \times 24$$

$$= \frac{1 \times \overset{4}{\cancel{24}}}{\cancel{6} \times 1}$$

$$= 4$$

Check: $6 \times 4 = 24$

In the example $1\frac{1}{2} \div 3$, we could write $3 \times ? = 1\frac{1}{2}$. The number we seek must be one-third of $1\frac{1}{2}$. Thus we can do the division by taking one-third of $1\frac{1}{2}$; that is, we multiply $1\frac{1}{2}$ by the reciprocal of 3.

$$1\frac{1}{2} \div 3 = 1\frac{1}{2} \times \frac{1}{3}$$

$$= \frac{\cancel{3}}{2} \times \frac{1}{\cancel{3}}$$

$$= \frac{1}{2}$$

Check: $3 \times \frac{1}{2} = \frac{3}{2} = 1\frac{1}{2}$

The rule for division by the reciprocal method is: Multiply the dividend by the reciprocal of the divisor. This is sometimes stated in short form as follows: Invert the divisor and multiply.

The following examples of cases that arise in division with fractions will be solved by both the reciprocal method and the common denominator method. The common denominator method more clearly shows the division process and is easier for the beginner to grasp. The reciprocal method is more obscure as to the reason for its use but has the advantage of speed and the possibility of cancellation of like factors, which simplifies the computation. It is the suggested method once the principles become familiar.

EXAMPLE: $\frac{2}{5} \div 4 = ?$

Common Denominator Method	Reciprocal Method
$\frac{2}{5} \div 4 = \frac{2}{5} \div \frac{20}{5}$	$\frac{2}{5} \div 4 = \frac{2}{5} \times \frac{1}{4}$
$= 2 \div 20$	$= \frac{\cancel{2} \times 1}{5 \times \underset{2}{\cancel{4}}}$
$= \frac{2}{20} = \frac{1}{10}$	$= \frac{1}{10}$

EXAMPLE: $2\frac{2}{3} \div 3 = ?$

Common Denominator Method	Reciprocal Method
$2\frac{2}{3} \div 3 = \frac{8}{3} \div \frac{9}{3}$	$2\frac{2}{3} \div 3 = \frac{8}{3} \times \frac{1}{3}$
$= 8 \div 9$	$= \frac{8 \times 1}{3 \times 3}$
$= \frac{8}{9}$	$= \frac{8}{9}$

EXAMPLE: $9 \div \frac{2}{7} = ?$

Common Denominator Method	Reciprocal Method
$9 \div \frac{2}{7} = \frac{63}{7} \div \frac{2}{7}$	$9 \div \frac{2}{7} = 9 \times \frac{7}{2}$
$= 63 \div 2$	$= \frac{9 \times 7}{1 \times 2}$
$= \frac{63}{2} = 31\frac{1}{2}$	$= \frac{63}{2} = 31\frac{1}{2}$

EXAMPLE: $10 \div 5\frac{3}{4} = ?$

Common Denominator Method	Reciprocal Method
$10 \div 5\frac{3}{4} = \frac{40}{4} \div \frac{23}{4}$	$10 \div 5\frac{3}{4} = 10 \times \frac{4}{23}$
$= 40 \div 23$	$= \frac{10 \times 4}{1 \times 23}$
$= \frac{40}{23} = 1\frac{17}{23}$	$= \frac{40}{23} = 1\frac{17}{23}$

EXAMPLE: $\frac{2}{3} \div \frac{1}{4} = ?$

Common Denominator Method	Reciprocal Method
$\frac{2}{3} \div \frac{1}{4} = \frac{8}{12} \div \frac{3}{12}$	$\frac{2}{3} \div \frac{1}{4} = \frac{2}{3} \times \frac{4}{1}$
$= 8 \div 3$	$= \frac{8}{3} = 2\frac{2}{3}$
$= \frac{8}{3} = 2\frac{2}{3}$	

EXAMPLE: $\frac{9}{16} \div \frac{3}{10} = ?$

Common Denominator Method	Reciprocal Method
$\frac{9}{16} \div \frac{3}{10} = \frac{45}{80} \div \frac{24}{80}$	$\frac{9}{16} \div \frac{3}{10} = \frac{9}{16} \times \frac{10}{3}$
$= 45 \div 24$	$= \frac{\overset{3}{\cancel{9}}}{\underset{8}{\cancel{16}}} \times \frac{\overset{5}{\cancel{10}}}{\cancel{3}}$
$= \frac{45}{24} = \frac{15}{8}$	
$= 1\frac{7}{8}$	$= \frac{15}{8} = 1\frac{7}{8}$

Practice problems. Perform the following division by the reciprocal method:

1. $\frac{3}{8} \div \frac{2}{3}$ 2. $2\frac{1}{3} \div 1\frac{1}{2}$ 3. $\frac{5}{8} \div \frac{5}{16}$ 4. $\frac{1}{3} \div \frac{4}{6}$

Answers:

1. $\frac{9}{16}$ 2. $1\frac{5}{9}$ 3. 2 4. $\frac{1}{2}$

COMPLEX FRACTIONS

When the numerator or denominator, or both, in a fraction are themselves composed of fractions, the resulting expression is called a complex fraction. The following expression is a complex fraction:

$$\frac{3/5}{3/4}$$

This should be read "three-fifths over three-fourths" or "three-fifths divided by three-fourths." Any complex fraction may be simplified by writing it as a division problem, as follows:

$$\frac{3/5}{3/4} = \frac{3}{5} \div \frac{3}{4}$$

$$= \frac{\cancel{3}}{5} \cdot \frac{4}{\cancel{3}}$$

$$= 4/5$$

Similarly,

$$\frac{3\frac{1}{3}}{2\frac{1}{2}} = \frac{10}{3} \div \frac{5}{2} = \frac{\overset{2}{\cancel{10}}}{3} \times \frac{2}{\cancel{5}} = \frac{4}{3} = 1\frac{1}{3}$$

Complex fractions may also contain an indicated operation in the numerator or denominator or both. Thus,

$$\frac{\frac{1}{2} + \frac{1}{3}}{\frac{9}{5} + \frac{1}{5}}$$

is a complex fraction. To simplify such a fraction we simplify the numerator and denominator and proceed as follows:

$$\frac{\frac{1}{2} + \frac{1}{3}}{\frac{9}{5} + \frac{1}{5}} = \frac{\frac{3}{6} + \frac{2}{6}}{\frac{10}{5}} = \frac{\frac{5}{6}}{2}$$

$$= \frac{5}{6} \div \frac{2}{1}$$

$$= \frac{5}{6} \times \frac{1}{2}$$

$$= \frac{5}{12}$$

Mixed numbers appearing in complex fractions usually show the plus sign.

Thus,

$$4\frac{2}{5} \div 7\frac{1}{3}$$

might be written

$$\frac{4 + \frac{2}{5}}{7 + \frac{1}{3}}$$

Practice problems. Simplify the following complex fractions:

1. $\dfrac{1}{\frac{3}{8}}$ 2. $\dfrac{2\frac{1}{2}}{3}$ 3. $\dfrac{3\frac{2}{3}}{2\frac{2}{5}}$ 4. $\dfrac{\frac{1}{4} + \frac{1}{3}}{\frac{1}{16} - \frac{1}{32}}$

Answers:

1. $\frac{8}{3}$ 2. $\frac{5}{6}$ 3. $1\frac{19}{36}$ 4. $18\frac{2}{3}$

Complex fractions may arise in electronics when it is necessary to find the total resistance of several resistances in parallel as shown in figure 4-8. The rule is: The total resistance of a parallel circuit is 1 divided by the sum of the reciprocals of the separate resistances. Written as a formula, this produces the following expression:

$$R_t = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

EXAMPLE: Find the total resistance of the parallel circuit in figure 4-8 (A). Substituting the values 3, 4, and 6 for the letters R_1, R_2, and R_3, we have the following:

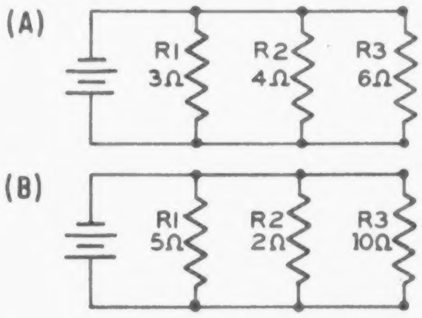


Figure 4-8.—Application of complex fractions in calculating electrical resistance.

$$R_t = \frac{1}{\frac{1}{3} + \frac{1}{4} + \frac{1}{6}}$$

The LCD of the fractions $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ is 12.

Thus,

$$R_t = \frac{1}{\frac{4}{12} + \frac{3}{12} + \frac{2}{12}}$$

$$= \frac{1}{\frac{9}{12}}$$

$$= \frac{12}{9} = \frac{4}{3}$$

$= 1\frac{1}{3}$ ohms (measure of resistance).

Practice problem: Find the total resistance of the parallel circuit in figure 4-8 (B).

Answer: $1\frac{1}{4}$ ohms.

CHAPTER 5

DECIMALS

The origin and meaning of the word "decimal" were discussed in chapter 1 of this course. Also discussed in chapter 1 were the concept of place value and the use of the number ten as the base for our number system. Another term which is frequently used to denote the base of a number system is RADIX. For example, two is the radix of the binary system and ten is the radix of the decimal system. The radix of a number system is always equal to the number of different digits used in the system. For example, the decimal system, with radix ten, has ten digits: 0 through 9.

DECIMAL FRACTIONS

A decimal fraction is a fraction whose denominator is 10 or some power of 10, such as 100, 1,000, or 10,000. Thus, $\frac{7}{10}$, $\frac{12}{100}$, and $\frac{215}{1000}$ are decimal fractions. Decimal fractions have special characteristics that make computation much simpler than with other fractions.

Decimal fractions complete our decimal system of numbers. In the study of whole numbers, we found that we could proceed to the left from the units place, tens, hundreds, thousands, and on indefinitely to any larger place value, but the development stopped with the units place. Decimal fractions complete the development so that we can proceed to the right of the units place to any smaller number indefinitely.

Figure 5-1 (A) shows how decimal fractions complete the system. It should be noted that as we proceed from left to right, the value of each place is one-tenth the value of the preceding place, and that the system continues uninterrupted with the decimal fractions.

Figure 5-1 (B) shows the system again, this time using numbers. Notice in (A) and (B) that the units place is the center of the system and that the place values proceed to the right or left of it by powers of ten. Ten on the left is balanced by tenths on the right, hundreds by hundredths, thousands by thousandths, etc.

Notice that 1/10 is one place to the right of the units digit, 1/100 is two places to the right, etc. (See fig. 5-1.) If a marker is placed after the units digit, we can decide whether a decimal digit is in the tenths, hundredths, or thousandths position by counting places to the right of the marker. In some European countries, the marker is a comma; but in the English-speaking countries, the marker is the DECIMAL POINT.

Thus, $\frac{3}{10}$ is written 0.3. To write $\frac{3}{100}$ it is necessary to show that 3 is in the second place to the right of the decimal point, so a zero is inserted in the first place. Thus, $\frac{3}{100}$ is written 0.03. Similarly, $\frac{3}{1000}$ can be written by inserting zeros in the first two places to the right of the decimal point. Thus, $\frac{3}{1000}$ is written 0.003. In the number 0.3, we say that 3 is in the first decimal place; in 0.03, 3 is in the second decimal place; and in 0.003, 3 is in the third decimal place. Quiet frequently decimal fractions are simply called decimals when written in this shortened form.

WRITING DECIMALS

Any decimal fraction may be written in the shortened form by a simple mechanical process. Simply begin at the right-hand digit of the numerator and count off to the left as many places as there are zeros in the denominator. Place the decimal point to the left of the last digit counted. The denominator may then be disregarded. If there are not enough digits, as many place-holding zeros as are necessary are added to the left of the left-hand digit in the numerator.

Thus, in $\frac{23}{10000}$, beginning with the digit 3, we count off four places to the left, adding two 0's as we count, and place the decimal point to the extreme left. (See fig. 5-2.) Either form is read "twenty-three ten-thousandths."

When a decimal fraction is written in the shortened form, there will always be as many decimal places in the shortened form as there

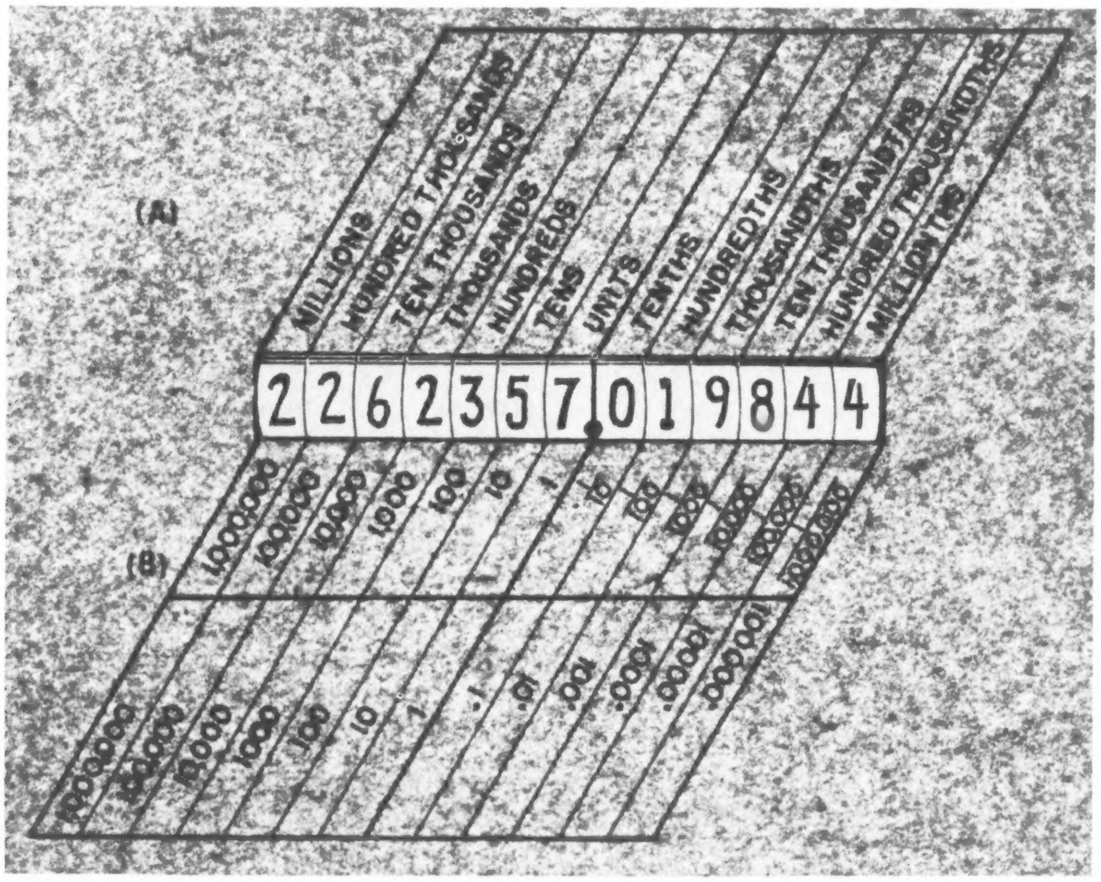

(A)
MILLIONS
HUNDRED THOUSANDS
TEN THOUSANDS
THOUSANDS
HUNDREDS
TENS
UNITS
TENTHS
HUNDREDTHS
THOUSANDTHS
TEN THOUSANDTHS
HUNDRED THOUSANDTHS
MILLIONTHS
2262357.019844
(B)
1,000,000
100,000
10,000
1000
100
10
1
1/10
1/100
1/1000
1/10000
1/100000
1/1000000
1000000
100000
10000
1000
100
10
1
.1
.01
.001
.0001
.00001
.000001

Figure 5-1.—Place values including decimals.

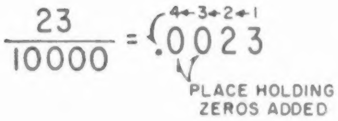

23/10000 = .0023
4←3←2←1
PLACE HOLDING ZEROS ADDED

Figure 5-2.—Conversion of a decimal fraction to shortened form.

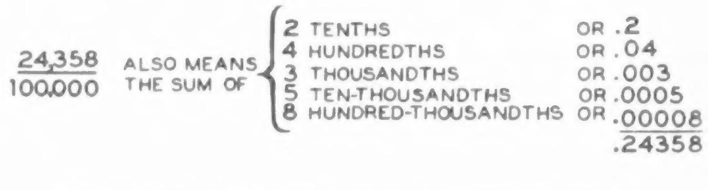

24,358/100,000
ALSO MEANS THE SUM OF
2 TENTHS OR .2
4 HUNDREDTHS OR .04
3 THOUSANDTHS OR .003
5 TEN-THOUSANDTHS OR .0005
8 HUNDRED-THOUSANDTHS OR .00008
.24358

Figure 5-3.—Steps in the conversion of a decimal fraction to shortened form.

are zeros in the denominator of the fractional form.

Figure 5-3 shows the fraction $\frac{24358}{100000}$ and what is meant when it is changed to the shortened form. This figure is presented to show further that each digit of a decimal fraction holds a certain position in the digit sequence and has a particular value.

By the fundamental rule of fractions, it should be clear that $\frac{5}{10} = \frac{50}{100} = \frac{500}{1000}$. Writing the same values in the shortened way, we have 0.5 = 0.50 = 0.500. In other words, the value of a decimal is not changed by annexing zeros at the right-hand end of the number. This is not true of whole numbers. Thus, 0.3, 0.30, an 0.300 are equal but 3, 30, and 300 are not equal Also notice that zeros directly after the deci mal point do change values. Thus 0.3 is no equal to either 0.03 or 0.003.

Decimals such as 0.125 are frequently seen Although the 0 on the left of the decimal poin is not required, it is often helpful. This is par ticularly true in an expression such as 32 ÷ 0.1 In this expression, the lower dot of the divisio symbol must not be crowded against the decima point; the 0 serves as an effective spacer. I any doubt exists concerning the clarity of ar expression such as .125, it should be written a 0.125.

Practice problems. In problems 1 through 4, change the fractions to decimals. In problems 5 through 8, write the given numbers as decimals:

1. 8/100
2. 5/1000
3. 43/1000
4. 32/10000
5. Four hundredths
6. Four thousandths
7. Five hundred one ten-thousandths
8. Ninety-seven thousandths

Answers:

1. 0.08
2. 0.005
3. 0.043
4. 0.0032
5. 0.04
6. 0.004
7. 0.0501
8. 0.097

READING DECIMALS

To read a decimal fraction in full, we read both its numerator and denominator, as in reading common fractions. To read 0.305, we read "three hundred five thousandths." The denominator is always 1 with as many zeros as decimal places. Thus the denominator for 0.14 is 1 with two zeros, or 100. For 0.003 it is 1,000; for 0.101 it is 1,000; and for 0.3 it is 10. The denominator may also be determined by counting off place values of the decimal. For 0.13 we may think "tenths, hundredths" and the fraction is in hundredths. In the example 0.1276 we may think "tenths, hundredths, thousandths, ten-thousandths." We see that the denominator is 10,000 and we read the fraction "one thousand two hundred seventy-six ten-thousandths."

A whole number with a fraction in the form of a decimal is called a MIXED DECIMAL. Mixed decimals are read in the same manner as mixed numbers. We read the whole number in the usual way followed by the word "and" and then read the decimal. Thus, 160.32 is read "one hundred sixty and thirty-two hundredths." The word "and" in this case, as with mixed numbers, means plus. The number 3.2 means three plus two tenths.

It is also possible to have a complex decimal. A COMPLEX DECIMAL contains a common fraction. The number $0.3\frac{1}{3}$ is a complex decimal and is read "three and one-third tenths." The number $0.87\frac{1}{2}$ means $87\frac{1}{2}$ hundredths. The common fraction in each case forms a part of the last or right-hand place.

In actual practice when numbers are called out for recording, the above procedure is not used. Instead, the digits are merely called out in order with the proper placing of the decimal point. For example, the number 216.003 is read, "two one six point zero zero three." The number 0.05 is read, "zero point zero five."

EQUIVALENT DECIMALS

Decimal fractions may be changed to equivalent fractions of higher or lower terms, as is the case with common fractions. If each decimal fraction is rewritten in its common fraction form, changing to higher terms is accomplished by multiplying both numerator and denominator by 10, or 100, or some higher power of 10. For example, if we desire to change $\frac{5}{10}$ to hundredths, we may do so by multiplying both numerator and denominator by 10. Thus,

$$\frac{5}{10} = \frac{50}{100}$$

In the decimal form, the same thing may be accomplished by simply annexing a zero. Thus,

$$0.5 = 0.50$$

Annexing a 0 on a decimal has the same effect as multiplying the common fraction form of the decimal by 10/10. This is an application of the fundamental rule of fractions. Annexing two 0's has the same effect as multiplying the common fraction form of the decimal by 100/100; annexing three 0's has the same effect as multiplying by 1000/1000; etc.

REDUCTION TO LOWER TERMS

Reducing to lower terms is known as ROUND-OFF, or simply ROUNDING, when dealing with decimal fractions. If it is desired to reduce 6.3000 to lower terms, we may simply drop as many end zeros as necessary since this is equivalent to dividing both terms of the fraction by some power of ten. Thus, we see that 6.3000 is the same as 6.300, 6.30, or 6.3.

It is frequently necessary to reduce a number such as 6.427 to some lesser degree of precision. For example, suppose that 6.427 is to be rounded to the nearest hundredth. The question to be decided is whether 6.427 is closer to 6.42 or 6.43. The best way to decide this question is to compare the fractions 420/1000, 427/1000, and 430/1000. It is obvious that 427/1000 is closer to 430/1000, and 430/1000 is equivalent to 43/100; therefore we say that 6.427, correct to the nearest hundredth, is 6.43.

A mechanical rule for rounding off can be developed from the foregoing analysis. Since the digit in the tenths place is not affected when we round 6.427 to hundredths, we may limit our attention to the digits in the hundredths and thousandths places. Thus the decision reduces to the question whether 27 is closer to 20 or 30. Noting that 25 is halfway between 20 and 30, it is clear that anything greater than 25 is closer to 30 than it is to 20.

In any number between 20 and 30, if the digit in the thousandths place is greater than 5, then the number formed by the hundredths and thousandths digits is greater than 25. Thus we would round the 27 in our original problem to 30, as far as the hundredths and thousandths digits are concerned. This result could be summarized as follows: When rounding to hundredths, if the digit in the thousandths place is greater than 5, increase the digit in the hundredths place by 1 and drop the digit in the thousandths place.

The digit in the thousandths place may be any one of the ten digits, 0 through 9. If these ten digits are split into two groups, one composed of the five smaller digits (0 through 4) and the other composed of the five larger digits, then 5 is counted as one of the larger digits. Therefore, the general rule for rounding off is stated as follows: If the digit in the decimal place to be eliminated is 5 or greater, increase the digit in the next decimal place to the left by 1. If the digit to be eliminated is less than 5, leave the retained digits unchanged.

The following examples illustrate the rule for rounding off:

1. 0.1414 rounded to thousandths is 0.141.
2. 3.147 rounded to tenths is 3.1.
3. 475 rounded to the nearest hundred is 500.

Observe carefully that the answer to example 2 is not 3.2. Some trainees make the error of treating the rounding process as a kind of chain reaction, in which one first rounds 3.147 to 3.15 and then rounds 3.15 to 3.2. The error of this method is apparent when we note that 147/1000 is closer to 100/1000 than it is to 200/1000.

Problems of the following type are sometimes confusing: Reduce 2.998 to the nearest hundredth. To drop the end figure we must increase the next figure by 1. The final result is 3.00. We retain the zeros to show that the answer is carried to the nearest hundredth.

Practice problems. Round off as indicated:

1. 0.5862 to hundredths
2. 0.345 to tenths
3. 2346 to hundreds
4. 3.999 to hundredths

Answers:

1. 0.59
2. 0.3
3. 2300
4. 4.00

CHANGING DECIMALS TO COMMON FRACTIONS

Any decimal may be reduced to a common fraction. To do this we simply write out the numerator and denominator in full and reduce to lowest terms. For example, to change 0.12 to a common fraction, we simply write out the fraction in full,

$$\frac{12}{100}$$

and reduce to lowest terms,

$$\frac{\overset{3}{\cancel{12}}}{\underset{25}{1\cancel{00}}} = \frac{3}{25}$$

Likewise, 0.77 is written

$$\frac{77}{100}$$

but this is in lowest terms so the fraction cannot be further reduced.

One way of checking to see if a decimal fraction can be reduced to lower terms is to consider the makeup of the decimal denominator. The denominator is always 10 or a power of 10. Inspection shows that the prime factors of 10 are 5 and 2. Thus, the numerator must be divisible by 5 or 2 or both, or the fraction cannot be reduced.

EXAMPLE: Change the decimal 0.0625 to a common fraction and reduce to lowest terms.

SOLUTION: $$0.0625 = \frac{625}{10000}$$

$$= \frac{625 \div 25}{10000 \div 25} = \frac{25}{400}$$

$$= \frac{1}{16}$$

Complex decimals are changed to common fractions by first writing out the numerator and denominator in full and then reducing the resulting complex fraction in the usual way. For example, to reduce $0.12\frac{1}{2}$, we first write

$$\frac{12\frac{1}{2}}{100}$$

Writing the numerator as an improper fraction we have

$$\frac{\frac{25}{2}}{100}$$

and applying the reciprocal method of division, we have

$$\frac{25}{2} \times \frac{1}{\underset{4}{\cancel{100}}} = \frac{1}{8}$$

Practice problems. Change the following decimals to common fractions in lowest terms:

1. 0.25
2. 0.375
3. $0.6\frac{1}{4}$
4. $0.03\frac{1}{5}$

Answers:

1. 1/4
2. 3/8
3. 5/8
4. 4/125

CHANGING COMMON FRACTIONS TO DECIMALS

The only difference between a decimal fraction and a common fraction is that the decimal fraction has 1 with a certain number of zeros (in other words, a power of 10) for a denominator. Thus, a common fraction can be changed to a decimal if it can be reduced to a fraction having a power of 10 for a denominator.

If the denominator of the common fraction in its lowest terms is made up of the prime factors 2 or 5 or both, the fraction can be converted to an exact decimal. If some other prime factor is present, the fraction cannot be converted exactly. The truth of this is evident when we consider the denominator of the new fraction. It must always be 10 or a power of 10, and we know the factors of such a number are always 2's and 5's.

The method of converting a common fraction to a decimal is illustrated as follows:

EXAMPLE: Convert 3/4 to a decimal.

SOLUTION: $$\frac{3}{4} = \frac{300}{400}$$

$$= \frac{300}{4} \times \frac{1}{100}$$

$$= 75 \times \frac{1}{100}$$

$$= 0.75$$

Notice that the original fraction could have been rewritten as 3000/4000, in which case the result would have been 0.750. On the other hand, if the original fraction had been rewritten as 30/40, the resulting division of 4 into 30 would not have been possible without a remainder. When the denominator in the original fraction has only 2's and 5's as factors, so that we know a remainder is not necessary, the fraction should be rewritten with enough 0's to complete the division with no remainder.

Observation of the results in the foregoing example leads to a shortcut in the conversion method. Noting that the factor 1/100 ultimately enters the answer in the form of a decimal, we could introduce the decimal point as the final step without ever writing the fraction 1/100. Thus the rule for changing fractions to decimals is as follows:

1. Annex enough 0's to the numerator of the original fraction so that the division will be exact (no remainder).
2. Divide the original denominator into the new numerator formed by annexing the 0's.
3. Place the decimal point in the answer so that the number of decimal places in the answer is the same as the number of 0's annexed to the original numerator.

If a mixed number in common fraction form is to be converted, convert only the fractional part and then write the two parts together. This is illustrated as follows:

$$2\frac{3}{4} = 2 + \frac{3}{4} = 2 + .75 = 2.75$$

Practice problems. Convert the following common fractions and mixed numbers to decimal form:

1. $\frac{1}{4}$ 2. $\frac{3}{8}$ 3. $\frac{5}{32}$ 4. $2\frac{5}{16}$

Answers:

1. 0.25 2. 0.375 3. 0.15625 4. 2.3125

Nonterminating Decimals

As stated previously, if the denominator of a common fraction contains some prime factor other than 2 or 5, the fraction cannot be converted completely to a decimal. When such fractions are converted according to the foregoing rule, the decimal resulting will never terminate. Consider the fraction 1/3. Applying the rule, we have

$$\begin{array}{r} .333\ \ldots \\ 3\overline{)1.0000} \\ \underline{9} \\ 10 \\ \underline{9} \\ 10 \\ \underline{9} \end{array}$$

The division will continue indefinitely. Any common fraction that cannot be converted exactly yields a decimal that will never terminate and in which the digits sooner or later recur. In the previous example, the recurring digit was 3. In the fraction 5/11, we have

$$\begin{array}{r} .4545 \\ 11\overline{)5.0000} \\ \underline{4\,4} \\ 60 \\ \underline{55} \\ 50 \\ \underline{44} \\ 60 \\ \underline{55} \end{array}$$

The recurring digits are 4 and 5.

When a common fraction generates such a repeating decimal, it becomes necessary to arbitrarily select a point at which to cease the repetition. This may be done in two ways. We may write the decimal fraction by rounding off at the desired point. For example, to round off the decimal generated by $\frac{1}{3}$ to hundredths, we carry the division to thousandths, see that this figure is less than 5, and drop it. Thus, $\frac{1}{3}$ rounded to hundredths is 0.33. The other method is to carry the division to the desired number of decimal places and carry the remaining incomplete division as a common fraction—that is, we write the result of a complex decimal. For example, $\frac{1}{3}$ carried to thousandths would be

$$\frac{1}{3} = \begin{array}{r} .333\frac{1}{3} \\ 3\overline{)1.000} \\ \underline{9} \\ 10 \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \end{array}$$

Practice problems. Change the following common fractions to decimals with three places and carry the incomplete division as a common fraction:

1. $\frac{7}{13}$ 2. $\frac{5}{9}$ 3. $\frac{4}{15}$ 4. $\frac{5}{12}$

Answers:

1. $0.538\frac{6}{13}$ 3. $0.266\frac{2}{3}$

2. $0.555\frac{5}{9}$ 4. $0.416\frac{2}{3}$

OPERATION WITH DECIMALS

In the study of addition of whole numbers, it was established that units must be added to units, tens to tens, hundreds to hundreds, etc. For convenience, in adding several numbers, units were written under units, tens under tens, etc. The addition of decimals is accomplished in the same manner.

ADDITION

In adding decimals, tenths are written under tenths, hundredths under hundredths, etc. When this is done, the decimal points fall in a straight line. The addition is the same as in adding whole numbers. Consider the following example:

$$\begin{array}{r} 2.18 \\ 34.35 \\ 0.14 \\ \underline{4.90} \\ 41.57 \end{array}$$

Adding the first column on the right gives 17 hundredths or 1 tenth and 7 hundredths. As with whole numbers, we write the 7 under the hundredths column and add the 1 tenth in the tenths column—that is, the column of the next higher order. The sum of the tenths column is 15 tenths or 1 unit and 5 tenths. The 5 is written under the tenths column and the 1 is added in the units column.

It is evident that if the decimal points are kept in a straight line—that is, if the place values are kept in the proper columns—addition with decimals may be accomplished in the ordinary manner of addition of whole numbers. It should also be noted that the decimal point of the sum falls directly under the decimal points of the addends.

SUBTRACTION

Subtraction of decimals likewise involves no new principles. Notice that the place values of the subtrahend in the following example are fixed directly under the corresponding place values in the minuend. Notice also that this causes the decimal points to be alined and that the figures in the difference (answer) also retain the correct columnar alinement.

$$\begin{array}{r} 45.76 \\ \underline{-31.87} \\ 13.89 \end{array}$$

We subtract column by column, as with whole numbers, beginning at the right.

Practice problems. Add or subtract as indicated:

1. 12.3 + 2.13 + 4 + 1.234
2. 0.5 + 0.04 + 12.001 + 10
3. 237.5 - 217.9
4. 9.04 - 7.156

Answers:

1. 19.664
2. 22.541
3. 19.6
4. 1.884

MULTIPLICATION

Multiplication of a decimal by a whole number may be explained by expressing the decimal as a fraction.

EXAMPLE: Multiply 6.12 by 4.

SOLUTION: $\frac{4}{1} \times \frac{612}{100} = \frac{2448}{100}$

$= 24.48$

When we perform the multiplication keeping the decimal form, we have

$$\begin{array}{r} 6.12 \\ \underline{4} \\ 24.48 \end{array}$$

By common sense, it is apparent that the whole number 4 times the whole number 6, with some fraction, will yield a number in the neighborhood of 24. Hence, the placing of the decimal point is reasonable.

An examination of several examples will reveal that the product of a decimal and a whole number has the same number of decimal places as the factor containing the decimal. Zeros, if any, at the end of the decimal should be rejected.

Multiplication of Two Decimals

To show the rule for multiplying two decimals together, we multiply the decimal in fractional form first and then in the conventional way, as in the following example:

$$0.4 \times 0.37$$

Writing these decimals as common fractions, we have

$$\frac{4}{10} \times \frac{37}{100} = \frac{4 \times 37}{10 \times 100}$$

$$= \frac{148}{1000}$$

$$= 0.148$$

In decimal form the problem is

$$\begin{array}{r} 0.37 \\ \underline{0.4} \\ 0.148 \end{array}$$

The placing of the decimal point is reasonable, since 4 tenths of 37 hundredths is a little less than half of 37 hundredths, or about 15 hundredths.

Consider the following example:

$$4.316 \times 3.4$$

In the common fraction form, we have

$$\begin{aligned} \frac{4316}{1000} \times \frac{34}{10} &= \frac{4316 \times 34}{1000 \times 10} \\ &= \frac{146744}{10000} \\ &= 14.6744 \end{aligned}$$

In the decimal form the problem is

$$\begin{array}{r} 4.316 \\ \underline{3.4} \\ 17264 \\ \underline{12948} \\ 14.6744 \end{array}$$

We note that 4 and a fraction times 3 and a fraction yields a product in the neighborhood of 12. Thus, the decimal point is in the logical place.

In the above examples it should be noted in each case that when we multiply the decimals together we are multiplying the numerators. When we place the decimal point by adding the number of decimal places in the multiplier and multiplicand, we are in effect multiplying the denominators.

When the numbers multiplied together are thought of as the numerators, the decimal points may be temporarily disregarded and the numbers may be considered whole. This justifies the apparent disregard for place value in the multiplication of decimals. We see that the rule for multiplying decimals is only a modification of the rule for multiplying fractions.

To multiply numbers in which one or more of the factors contain a decimal, multiply as though the numbers were whole numbers. Mark off as many decimal places in the product as there are decimal places in the factors together.

Practice problems. Multiply as indicated:

1. 3.7 x 0.02 2. 0.45 x 0.7

3. $\begin{array}{r} 6.5 \\ \underline{\times 0.01} \end{array}$ 4. $\begin{array}{r} 0.0073 \\ \underline{\times 5.4} \end{array}$

Answers:

1. 0.074 2. 0.315

3. 0.065 4. 0.03942

Multiplying by Powers of 10

Multiplying by a power of 10 (10, 100, 1,000, etc.) is done mechanically by simply moving the decimal point to the right as many places as there are zeros in the multiplier. For example, 0.00687 is multiplied by 1,000 by moving the decimal point three places to the right as follows:

$$1{,}000 \times 0.00687 = 6.87$$

Multiplying a number by 0.1, 0.01, 0.001, etc., is done mechanically by simply moving the decimal point to the left as many places as there are decimal places in the multiplier. For example, 348.2 is multiplied by 0.001 by moving the decimal point three places to the left as follows:

$$348.2 \times 0.001 = 0.3482$$

DIVISION

When the dividend is a whole number, we recognize the problem of division as that of converting a common fraction to a decimal. Thus in the example 5 ÷ 8, we recall that the problem could be written

$$\begin{aligned} \frac{5000}{1000} \div 8 &= \frac{5000 \div 8}{1000} \\ &= \frac{625}{1000} \\ &= .625 \end{aligned}$$

This same problem may be worked by the following, more direct method:

$$\frac{5}{8} = \begin{array}{r} .625 \\ 8\overline{)5.000} \\ \underline{4\ 8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \end{array}$$

Since not all decimals generated by division terminate so early as that in the above example, if at all, it should be predetermined as to how many decimal places it is desired to carry the quotient. If it is decided to terminate a quotient at the third decimal place, the division should be carried to the fourth place so that the correct rounding off to the third place may be determined.

When the dividend contains a decimal, the same procedure applies as when the dividend is whole. Notice the following examples (rounded to three decimal places):

```
1. 6.31 ÷ 8                     .7887 = .789
                              8/6.3100
                                5 6
                                 71
                                 64
                                  70
                                  64
                                   60
                                   56
                                    4

2. 0.0288 ÷ 32                  0.0009 = 0.001
                             32/0.0288
                                   288
```

Observe in each case (including the case where the dividend is whole), that the quotient contains the same number of decimal places as the number used in the dividend. Notice also that the place values are rigid; that is, tenths in the quotient appear over tenths in the dividend, hundredths over hundredths, etc.

Practice problems. In the following division problems, round off each quotient correct to three decimal places.

1. 10 ÷ 6
2. 23.5 ÷ 16
3. 2.743 ÷ 77
4. 1.00 ÷ 3

Answers:

1. 1.667
2. 1.469
3. 0.036
4. 0.333

Decimal Divisors

In the foregoing examples, the divisor in each case was an integer. Division with divisors which are decimals may be accomplished by changing the divisor and dividend so that the divisor becomes a whole number.

Recalling that every division expression may be written in fraction form, we use the fundamental rule of fractions as follows: Rewrite the division problem as a fraction. Multiply the numerator (dividend) and denominator (divisor) by 10, 100, or some higher power of 10; the power of 10 must be large enough to change the divisor to a whole number. This rule is illustrated as follows:

$$\begin{aligned} 2.568 \div 0.24 &= \frac{2.568}{0.24} \\ &= \frac{2.568}{0.24} \times \frac{100}{100} \\ &= \frac{256.8}{24} \end{aligned}$$

Thus 2.568 divided by 0.24 is the same as 256.8 divided by 24.

From the mechanical standpoint, the foregoing rule has the effect of moving the decimal point to the right, as many places as necessary to change the divisor to an integer. Therefore the rule is sometimes stated as follows: When the divisor is a decimal, change it to a whole number by moving the decimal point to the right. Balance the change in the divisor by moving the decimal point in the dividend an equal number of places to the right.

The following example illustrates this version of the rule:

```
         9 1.1
0.9∧/81.9∧9
```

The inverted v, called a caret, is used as a marker to indicate the new position of the decimal point. Notice that the decimal point in the quotient is placed immediately above the caret in the dividend. Alinement of the first quotient digit immediately above the 1 in the dividend, and the second quotient digit above the 9, assures that these digits are placed properly with respect to the decimal point.

Practice problems. In the following division problems, round off each quotient to three decimal places:

1. 0.02958 ÷ 0.12
2. 30.625 ÷ 3.5
3. 4610 ÷ 0.875
4. 0.000576 ÷ 0.008

Answers:

1. 0.247
2. 8.750
3. 5268.571
4. 0.072

Dividing by Powers of 10

Division of any number by 10, 100, 1,000, etc., is really just an exercise in placing the decimal point of a decimal fraction. Thus, 5,031 ÷ 100 may be thought of as the decimal fraction $\frac{5031}{100}$; to remove the denominator, we simply count off two places from the right. Thus,

$$\frac{5031}{100} = 50.31$$

The following three examples serve to illustrate this procedure further:

$$401 \div 10 = 40.1$$
$$2 \div 1{,}000 = .002$$
$$11{,}431 \div 100 = 114.31$$

If the dividend already contains a decimal part, begin counting with the first number to the left of the decimal point. Thus, 243.6 ÷ 100 = 2.436. When the decimal point is not shown in a number, it is always considered to be to the right of the right-hand digit.

Dividing by 0.1, 0.01, 0.001, etc., may also be accomplished by a simple mechanical rule. We simply begin at the position of the decimal point in the dividend and count off as many places to the right as there are decimal places in the divisor. The decimal point is then placed to the right of the last digit counted. If there are not enough digits, zeros may be added.

The foregoing rule is based on the fact that 0.1 is really $\frac{1}{10}$, 0.01 is $\frac{1}{100}$, 0.001 is $\frac{1}{1000}$, etc. For example,

$$\begin{aligned} 23 \div 0.1 &= 23 \div \frac{1}{10} \\ &= 23 \times \frac{10}{1} \\ &= 230 \end{aligned}$$

Notice that dividing by 0.1 is the same as multiplying by 10. Likewise,

$$\begin{aligned} 234.1 \div 0.001 &= 234.1 \div \frac{1}{1000} \\ &= 234.1 \times \frac{1000}{1} \\ &= 234{,}100 \end{aligned}$$

and

$$24 \div 0.01 = 24 \div \frac{1}{100} = 24 \times \frac{100}{1} = 2{,}400$$

Practice problems. Divide by relocation of the decimal point.

1. 276 ÷ 100
2. 2,845 ÷ 1,000
3. 276 ÷ 0.01
4. 2,845 ÷ 0.001

Answers:

1. 2.76
2. 2.845
3. 27,600
4. 2,845,000

CHAPTER 6

PERCENTAGE AND MEASUREMENT

In the discussion of decimal fractions, it was shown that for convenience in writing fractions whose denominators are 10 or some power of 10, the decimal point could be employed and the denominators could be dropped. Thus, this special group of fractions could be written in a much simpler way. As early as the 15th century, businessmen made use of certain decimal fractions so much that they gave them the special designation PERCENT.

MEANING OF PERCENT

The word "percent" is derived from Latin. It was originally "per centum," which means "by the hundred." Thus the statement is often made that "percent means hundredths."

Percentage deals with the group of decimal fractions whose denominators are 100—that is, fractions of two decimal places. Since hundredths were used so frequently, the decimal point was dropped and the symbol % was placed after the number and read "percent" (per 100). Thus, 0.15 and 15% represent the same value, 15/100. The first is read "15 hundredths," and the second is read "15 percent." Both mean 15 parts out of 100.

Ordinarily, percent is used in discussing relative values. For example, 25 percent may convey an idea of relative value or relationship. To say "25 percent of the crew is ashore" gives an idea of what part of the crew is gone, but it does not tell how many. For example, 25 percent of the crew would represent vastly different numbers if the comparison were made between an LSM and a cruiser. When it is necessary to use a percent in computation, the number is written in its decimal form to avoid confusion.

By converting all decimal fractions so that they had the common denominator 100, men found that they could mentally visualize the relative size of the part of the whole that was being considered.

CHANGING DECIMALS TO PERCENT

Since percent means hundredths, any decimal may be changed to percent by first expressing it as a fraction with 100 as the denominator. The numerator of the fraction thus formed indicates how many hundredths we have, and therefore it indicates "how many percent" we have. For example, 0.36 is the same as 36/100. Therefore, 0.36 expressed as a percentage would be 36 percent. By the same reasoning, since 0.052 is equal to 5.2/100, 0.052 is the same as 5.2 percent.

In actual practice, the step in which the denominator 100 occurs is seldom written down. The expression in terms of hundredths is converted mentally to percent. This results in the following rule: To change a decimal to percent, multiply the decimal by 100 and annex the percent sign (%). Since multiplying by 100 has the effect of moving the decimal point two places to the right, the rule is sometimes stated as follows: To change a decimal to percent, move the decimal point two places to the right and annex the percent sign.

Changing Common Fractions and Whole Numbers To Percent

Common fractions are changed to percent by first expressing them as decimals. For example, the fraction 1/4 is equivalent to the decimal 0.25. Thus 1/4 is the same as 25 percent.

Whole numbers may be considered as special types of decimals (for example, 4 may be written as 4.00) and thus may be expressed in terms of percentage. The meaning of an expression such as 400 percent is vague unless we keep in mind that percentage is a form of comparison. For example, a question which often arises is "How can I have more than 100 percent of something, if 100 percent means all of it?"

This question seems reasonable, if we limit our attention to such quantities as test scores. However, it is also reasonable to use percentage in comparing a current set of data with a previous set. For example, if the amount of electrical power used by a Navy facility this year is double the amount used last year, then this year's power usage is 200 percent of last year's usage.

The meaning of a phrase such as "200 percent of last year's usage" is often misinterpreted. A total amount that is 200 percent of the previous amount is not the same as an increase of 200 percent. The increase in this case is only 100 percent, for a total of 200. If the increase had been 200 percent, then the new usage figure would be 300 percent of the previous figure.

Baseball batting averages comprise a special case in which percentage is used with only occasional reference to the word "percent." The percentages in batting averages are expressed in their decimal form, with the figure 1.000 representing 100 percent. Although a batting average of 0.300 is referred to as "batting 300," this is actually erroneous nomenclature from the strictly mathematical standpoint. The correct statement, mathematically, would be "batting point three zero zero" or "batting 30 percent."

Practice problems. Change each of the following numbers to percent:

1. 0.0065
2. 1.25
3. 0.363
4. 3/4
5. 7
6. 1/2

Answers:

1. 0.65%
2. 125%
3. 36.3%
4. 75%
5. 700%
6. 50%

CHANGING A PERCENT TO A DECIMAL

Since we do not compute with numbers in the percent form, it is often necessary to change a percent back to the decimal form. The procedure is just opposite to that used in changing decimals to percents: To change a percent to a decimal, drop the percent sign and divide the number by 100. Mechanically, the decimal point is simply shifted two places to the left and the percent sign is dropped. For example, 25 percent is the same as the decimal 0.25. Percents larger than 100 percent are changed to decimals by the same procedure as ordinary percents. For example, 125 percent is equivalent to 1.25.

Practice problems. Change the following percents to decimals:

1. 2.5%
2. 0.63%
3. 125%
4. 25%
5. $5\frac{3}{4}\%$
6. $9\frac{1}{2}\%$

Answers:

1. 0.025
2. 0.0063
3. 1.25
4. 0.25
5. 5.75% = 0.0575
6. 9.50% = 0.095

THE THREE PERCENTAGE CASES

To explain the cases that arise in problems involving percents, it is necessary to define the terms that will be used. Rate (r) is the number of hundredths parts taken. This is the number followed by the percent sign. The base (b) is the whole on which the rate operates. Percentage (p) is the part of the base determined by the rate. In the example

$$5\% \text{ of } 40 = 2$$

5% is the rate, 40 is the base, and 2 is the percentage.

There are three cases that usually arise in dealing with percentage, as follows:

Case I—To find the percentage when the base and rate are known.

EXAMPLE: What number is 6% of 50?

Case II—To find the rate when the base and percentage are known.

EXAMPLE: 20 is what percent of 60?

Case III—To find the base when the percentage and rate are known.

EXAMPLE: The number 5 is 25% of what number?

Case I

In the example

$$6\% \text{ of } 50 = ?$$

the "of" has the same meaning as it does in fractional examples, such as

$$\frac{1}{4} \text{ of } 16 = ?$$

In other words, "of" means to multiply. Thus, to find the percentage, multiply the base by the rate. Of course the rate must be changed from a percent to a decimal before multiplying can be done. Rate times base equals percentage. Thus,

$$6\% \text{ of } 50 = ?$$
$$0.06 \times 50 = 3$$

The number that is 6% of 50 is 3.

FRACTIONAL PERCENTS.—A fractional percent represents a part of 1 percent. In a case such as this, it is sometimes easier to find 1 percent of the number and then find the fractional part. For example, we would find 1/4 percent of 840 as follows:

$$\begin{aligned} 1\% \text{ of } 840 &= 0.01 \times 840 \\ &= 8.40 \end{aligned}$$

$$\text{Therefore, } \tfrac{1}{4}\% \text{ of } 840 = 8.40 \times \tfrac{1}{4}$$
$$= 2.10$$

Case II

To explain case II and case III, we notice in the foregoing example that the base corresponds to the multiplicand, the rate corresponds to the multiplier, and the percentage corresponds to the product.

$$\begin{array}{rl} 50 & \text{(base or multiplicand)} \\ \underline{.06} & \text{(rate or multiplier)} \\ 3.00 & \text{(percentage or product)} \end{array}$$

Recalling that the product divided by one of its factors gives the other factor, we can solve the following problem:

$$?\% \text{ of } 60 = 20$$

We are given the base (60) and percentage (20).

$$\begin{array}{rl} 60 & \text{(base)} \\ \underline{?} & \text{(rate)} \\ 20 & \text{(percentage)} \end{array}$$

We then divide the product (percentage) by the multiplicand (base) to get the other factor (rate). Percentage divided by base equals rate. The rate is found as follows:

$$\begin{aligned} \frac{20}{60} &= \frac{1}{3} \\ &= .33\tfrac{1}{3} \\ &= 33\tfrac{1}{3}\% \text{ (rate)} \end{aligned}$$

The rule for case II, as illustrated in the foregoing problem, is as follows: To find the rate when the percentage and base are known, divide the percentage by the base. Write the quotient in the decimal form first, and finally as a percent.

Case III

The unknown factor in case III is the base, and the rate and percentage are known.

EXAMPLE: 25% of ? = 5

$$\begin{array}{rl} ? & \text{(base)} \\ \underline{.25} & \text{(rate)} \\ 5.00 & \text{(percentage)} \end{array}$$

We divide the product by its known factor to find the other factor. Percentage divided by rate equals base. Thus,

$$\frac{5}{.25} = 20 \text{ (base)}$$

The rule for case III may be stated as follows: To find the base when the rate and percentage are known, divide the percentage by the rate.

Practice problems. In each of the following problems, first determine which case is involved; then find the answer.

1. What is $\frac{3}{4}\%$ of 740?
2. 7.5% of 2.75 = ?
3. 8 is 2% of what number?
4. ?% of 18 = 15.
5. 12% of ? = 12.
6. 8 is what percent of 32?

Answers:

1. Case I; 5.55
2. Case I; 0.20625
3. Case III; 400
4. Case II; $83\frac{1}{3}\%$
5. Case III; 100
6. Case II; 25%

PRINCIPLES OF MEASUREMENT

Computation with decimals frequently involves the addition or subtraction of numbers which do not have the same number of decimal places. For example, we may be asked to add such numbers as 4.1 and 32.31582. How should they be added? Should zeros be annexed to 4.1 until it is of the same order as the other decimal (to the same number of places)? Or, should .31582 be rounded off to tenths? Would the sum be accurate to tenths or hundred-thousandths? The answers to these questions depend on how the numbers orignially arise.

Some decimals are finite or are considered as such because of their use. For instance, the decimal that represents $\frac{1}{2}$, that is 0.5, is as accurate at 0.5 as it is at 0.5000. Likewise, the decimal that represents $\frac{1}{8}$ has the value 0.125 and could be written just as accurately with additional end zeros. Such numbers are said to be finite. Counting numbers are finite. Dollars and cents are examples of finite values. Thus, $10.25 and $5.00 are finite values.

To add the decimals that represent $\frac{1}{8}$ and $\frac{1}{2}$, it is not necessary to round off 0.125 to tenths. Thus, 0.5 + 0.125 is added as follows:

$$\begin{array}{r} 0.500 \\ \underline{0.125} \\ 0.625 \end{array}$$

Notice that the end zeros were added to 0.5 to carry it out the same number of places as 0.125. It is not necessary to write such place-holding zeros if the figures are kept in the correct columns and decimal points are alined. Decimals that have a definite fixed value may be added or subtracted although they are of different order.

On the other hand, if the numbers result from measurement of some kind, then the question of how much to round off must be decided in terms of the precision and accuracy of the measurements.

ESTIMATION

Suppose that two numbers to be added resulted from measurement. Let us say that one number was measured with a ruler marked off in tenths of an inch and was found, to the nearest tenth of an inch, to be 2.3 inches. The other number measured with a precision rule was found, to the nearest thousandth of an inch, to be 1.426 inches.

Each of these measurements requires estimation between marks on the rule, and estimation between marks on any measuring instrument is subject to human error. Experience has shown that the best the average person can do with consistency is to decide whether a measurement is more or less than halfway between marks. The correct way to state this fact mathematically is to say that a measurement made with an instrument marked off in tenths of an inch involves a maximum probable error of 0.05 inch (five hundredths is one-half of one tenth). By the same reasoning, the probable error in a measurement made with an instrument marked in thousandths of an inch is 0.0005 inch.

PRECISION

In general, the probable error in any measurement is one-half the size of the smallest division on the measuring instrument. Thus the precision of a measurement depends upon how precisely the instrument is marked. It is important to realize that precision refers to the size of the smallest division on the scale; it has nothing to do with the accuracy (correctness) of the markings. In other words, to say that one instrument is more precise than another does not imply that the less precise instrument is poorly manufactured. In fact, it would be possible to make an instrument with very high apparent precision, and yet mark it carelessly so that measurements taken with it would be inaccurate.

From the mathematical standpoint, the precision of a number resulting from measurement depends upon the number of decimal places; that is, a larger number of decimal places means a smaller probable error. In 2.3 inches the probable error is 0.05 inch, since 2.3 actually lies somewhere between 2.25 and 2.35. In 1.426 inches there is a much smaller probable error of 0.0005 inch. If we add 2.300 + 1.426 and get an answer in thousandths, the answer, 3.726 inches, would appear to be precise to thousandths; but this is not true since there was a probable error of .05 in one of the addends. Also 2.300 appears to be precise to thousandths but in this example it is precise only to tenths. It is evident that the precision of a sum is no greater than the precision of the least precise addend. It can also be shown that the precision of a difference is no greater than the less precise number compared.

To add or subtract numbers of different orders, all numbers should first be rounded off to the order of the least precise number. In the foregoing example, 1.426 should be rounded to tenths—that is, 1.4.

This rule also applies to repeating decimals. Since it is possible to round off a repeating decimal at any desired point, the degree of precision desired should be determined and all repeating decimals to be added should be rounded to this level. Thus, to add the decimals generated by $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{5}{12}$ correct to thousandths, first round off each decimal to thousandths, and then add, as follows:

$$\begin{array}{r} .333 \\ .667 \\ .417 \\ \hline 1.417 \end{array}$$

When a common fraction is used in recording the results of measurement, the denominator of the fraction indicates the degree of precision. For example, a ruler marked in sixty-fourths of an inch has much smaller divisions than one marked in fourths of an inch. Therefore a measurement of $3\frac{4}{64}$ inches is more precise than a measure of $3\frac{1}{4}$ inches, even though the two fractions are numerically equal. Remember that a measurement of $3\frac{4}{64}$ inches contains a probable error of only one-half of one sixty-fourth of an inch. On the other hand, if the smallest division on the ruler is one-fourth of an inch, then a measurement of $3\frac{1}{4}$ inches contains a probable error of one-eighth of an inch.

ACCURACY

Even though a number may be very precise, which indicates that it was measured with an instrument having closely spaced divisions, it may not be very accurate. The accuracy of a measurement depends upon the relative size of the probable error when compared with the quantity being measured. For example, a distance of 25 yards on a pistol range may be measured carefully enough to be correct to the nearest inch. Since there are 900 inches in 25 yards, this measurement is between 899.5 inches and 900.5 inches. When compared with the total of 900 inches, the 0.5-inch probable error is not very great.

On the other hand, a length of pipe may be measured rather precisely and found to be 3.2 inches long. The probable error here is 0.05 inch, and this measurement is thus more precise than that of the pistol range mentioned before. To compare the accuracy of the two measurements, we note that 0.05 inch out of a total of 3.2 inches is the same as 0.5 inch out of 32 inches. Comparing this with the figure obtained in the other example (0.5 inch out of 900), we conclude that the more precise measurement is actually the less accurate of the two measurements considered.

It is important to realize that the location of the decimal point has no bearing on the accuracy of the number. For example, 1.25 dollars represents exactly the same amount of money as 125 cents. These are equally accurate ways of representing the same quantity, despite the fact that the decimal point is placed differently.

Practice problems. In each of the following problems, determine which number of each pair is more accurate and which is more precise:

1. 3.72 inches or 2,417 feet
2. 2.5 inches or 17.5 inches
3. $5\frac{3}{4}$ inches or $12\frac{7}{8}$ inches
4. 34.2 seconds or 13 seconds

Answers:

1. 3.72 inches is more precise.
   2,417 feet is more accurate.
2. The numbers are equally precise.
   17.5 inches is more accurate.
3. $12\frac{7}{8}$ inches is more precise and more accurate.
4. 34.2 seconds is more precise and more accurate.

Percent of Error

The accuracy of a measurement is determined by the RELATIVE ERROR. The relative error is the ratio between the probable error and the quantity being measured. This ratio is simply the fraction formed by using the probable error as the numerator and the measurement itself as the denominator. For example, suppose that a metal plate is found to be 5.4 inches long, correct to the nearest tenth of an inch. The maximum probable error is five hundredths of an inch (one-half of one tenth of an inch) and the relative error is found as follows:

$$\frac{\text{probable error}}{\text{measured value}} = \frac{0.05}{5.4}$$

$$= \frac{5}{540}$$

Thus the relative error is 5 parts out of 540.

Relative error is usually expressed as PERCENT OF ERROR. When the denominator of the fraction expressing the error ratio is divided into the numerator, a decimal is obtained. This decimal, converted to percent, gives the percent of error. For example, the error in the foregoing problem could be stated as 0.93 percent, since the ratio 5/540 reduces to 0.0093 (rounded off) in decimal form.

Significant Digits

The accuracy of a measurement is often described in terms of the number of significant digits used in expressing it. If the digits of a number resulting from measurement are examined one by one, beginning with the left-hand digit, the first digit that is not 0 is the first significant digit. For example, 2345 has four significant digits and 0.023 has only two significant digits.

The digits 2 and 3 in a measurement such as 0.023 inch signify how many thousandths of an inch comprise the measurement. The 0's are of no significance in specifying the number of thousandths in the measurement; their presence is required only as "place holders" in placing the decimal point.

A rule that is often used states that the significant digits in a number begin with the first nonzero digit (counting from left to right) and end with the last digit. This implies that 0 can be a significant digit if it is not the first digit in the number. For example, 0.205 inch is a measurement having three significant digits. The 0 between the 2 and the 5 is significant because it is a part of the number specifying how many thousandths are in the measurement.

The rule stated in the foregoing paragraph fails to classify final 0's on the right. For example, in a number such as 4,700, the number of significant digits might be two, three, or four. If the 0's merely locate the decimal point (that is, if they show the number to be approximately forty-seven hundred rather than forty seven), then the number of significant digits is two. However, if the number 4,700 represents a number such as 4,703 rounded off to the nearest hundred, there are three significant digits. The last 0 merely locates the decimal point. If the number 4,700 represents a number such as 4,700.4 rounded off, then the number of significant digits is four.

Unless we know how a particular number was measured, it is sometimes impossible to determine whether right-hand 0's are the result of rounding off. However, in a practical situation it is normally possible to obtain information concerning the instruments used and the degree of precision of the original data before any rounding was done.

In a number such as 49.30 inches, it is reasonable to assume that the 0 in the hundredths place would not have been recorded at all if it were not significant. In other words, the instrument used for the measurement can be read to the nearest hundredth of an inch. The 0 on the right is thus significant. This conclusion can be reached another way by observing that the 0 in 49.30 is not needed as a place holder in placing the decimal point. Therefore its presence must have some other significance.

The facts concerning significant digits may be summarized as follows:

1. Digits other than 0 are always significant.
2. Zero is significant when it falls between significant digits.
3. Any final 0 to the right of the decimal point is significant.
4. When a 0 is present only as a place holder for locating the decimal point, it is not significant.
5. The following categories comprise the significant digits of any measurement number:
    a. The first nonzero left-hand digit is significant.
    b. The digit which indicates the precision of the number is significant. This is the digit farthest to the right, except when the right-hand digit is 0. If it is 0, it may be only a place holder when the number is an integer.

c. All digits between significant digits are significant.

Practice problems. Determine the percent of error and the number of significant digits in each of the following measurements:

1. 5.4 feet
2. 0.00042 inch
3. 4.17 sec
4. 147.50 miles

Answers:

1. Percent of error: 0.93%
   Significant digits: 2
2. Percent of error: 1.19%
   Significant digits: 2
3. Percent of error: 0.12%
   Significant digits: 3
4. Percent of error: 0.0034%
   Significant digits: 5

CALCULATING WITH APPROXIMATE NUMBERS

The concepts of precision and accuracy form the basis for the rules which govern calculation with approximate numbers (numbers resulting from measurement).

Addition and Subtraction

A sum or difference can never be more precise than the least precise number in the calculation. Therefore, before adding or subtracting approximate numbers, they should be rounded to the same degree of precision. The more precise numbers are all rounded to the precision of the least precise number in the group to be combined. For example, the numbers 2.95, 32.7, and 1.414 would be rounded to tenths before adding as follows:

$$\begin{array}{r} 3.0 \\ 32.7 \\ \underline{1.4} \end{array}$$

Multiplication and Division

When two numbers are multiplied, the result often has several more digits than either of the original factors. Division also frequently produces more digits in the quotient than the original data possessed, if the division is "carried out" to several decimal places. Results such as these appear to have more significant digits than the original measurements from which they came, giving the false impression of greater accuracy than is justified. In order to correct this situation, the following rule is used:

In order to multiply or divide two approximate numbers having an equal number of significant digits, round the answer to the same number of significant digits as are shown in the original data. If one of the original factors has more significant digits than the other, round the more accurate number before multiplying. It should be rounded to one more significant digit than appears in the less accurate number; the extra digit protects the answer from the effects of multiple rounding. After performing the multiplication or division, round the result to the same number of significant digits as are shown in the less accurate of the original factors.

Practice problems:

1. Find the sum of the sides of a triangle in which the lengths of the three sides are as follows: 2.5 inches, 3.72 inches, and 4.996 inches.
2. Find the product of the length and width of a rectangle which is 2.95 feet long and 0.9046 foot wide.

Answers:

1. 11.2 inches
2. 2.67 square feet

MICROMETERS AND VERNIERS

Closely associated with the study of decimals is a measuring instrument known as a micrometer. The ordinary micrometer is capable of measuring accurately to one-thousandth of an inch. One-thousandth of an inch is about the thickness of a human hair or a thin sheet of paper. The parts of a micrometer are shown in figure 6-1.

MICROMETER SCALES

The spindle and the thimble move together. The end of the spindle (hidden from view in figure 6-1) is a screw with 40 threads per inch. Consequently, one complete turn of the thimble moves the spindle one-fortieth of an inch or

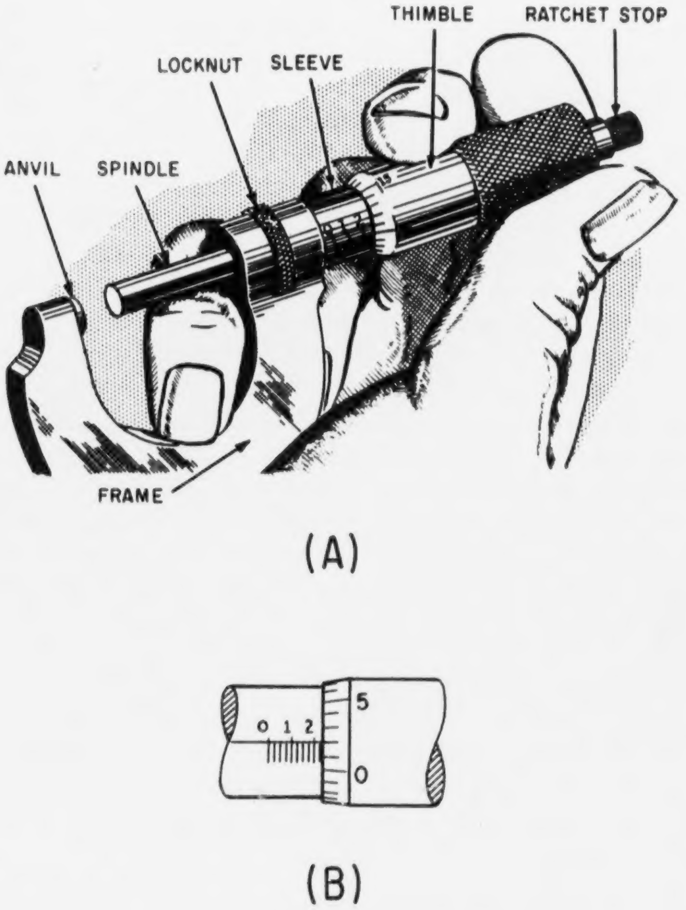


Figure 6-1.—(A) Parts of a micrometer; (B) micrometer scales.

0.025 inch since $\frac{1}{40}$ is equal to 0.025. The sleeve has 40 markings to the inch. Thus each space between the markings on the sleeve is also 0.025 inch. Since 4 such spaces are 0.1 inch (that is, 4 x 0.025), every fourth mark is labeled in tenths of an inch for convenience in reading. Thus, 4 marks equal 0.1 inch, 8 marks equal 0.2 inch, 12 marks equal 0.3 inch, etc.

To enable measurement of a partial turn, the beveled edge of the thimble is divided into 25 equal parts. Thus each marking on the thimble is $\frac{1}{25}$ of a complete turn, or $\frac{1}{25}$ of $\frac{1}{40}$ of an inch. Multiplying $\frac{1}{25}$ times 0.025 inch, we find that each marking on the thimble represents 0.001 inch.

READING THE MICROMETER

It is sometimes convenient when learning to read a micrometer to write down the component parts of the measurement as read on the scales and then to add them. For example, in figure 6-1 (B) there are two major divisions visible (0.2 inch). One minor division is showing clearly (0.025 inch). The marking on the thimble nearest the horizontal or index line of the sleeve is the second marking (0.002 inch). Adding these parts, we have

$$\begin{array}{r} 0.200 \\ 0.025 \\ 0.002 \\ \hline 0.227 \end{array}$$

Thus, the reading is 0.227 inch. As explained previously, this is read verbally as "two hundred twenty-seven thousandths." A more skillful method of reading the scales is to read all digits as thousandths directly and to do any adding mentally. Thus, we read the major division on the scale as "two hundred thousandths" and the minor division is added on mentally. The mental process for the above setting then would be "two hundred twenty-five; two hundred twenty-seven thousandths."

Practice problems:

1. Read each of the micrometer settings shown in figure 6-2.

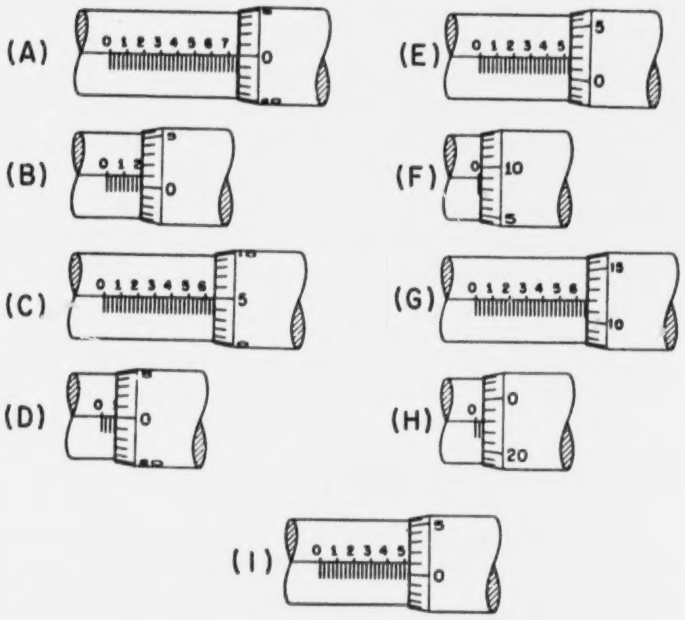


Figure 6-2.—Micrometer settings.

Answers:

1. (A) 0.750
   (B) 0.201
   (C) 0.655
   (D) 0.075
   (E) 0.527
   (F) 0.009
   (G) 0.662
   (H) 0.048
   (I) 0.526

VERNIER

Sometimes the marking on the thimble of the micrometer does not fall directly on the index line of the sleeve. To make possible readings even smaller than thousandths, an ingenious device is introduced in the form of an additional scale. This scale, called a VERNIER, was named after its inventor, Pierre Vernier. The vernier makes possible accurate readings to the ten-thousandth of an inch.

Principle of the Vernier

Suppose a ruler has markings every tenth of an inch but it is desired to read accurately to hundredths. A separate, freely sliding vernier scale (fig. 6-3) is added to the ruler. It has 10 markings on it that take up the same distance as 9 markings on the ruler scale. Thus, each space on the vernier is $\frac{1}{10}$ of $\frac{9}{10}$ inch, or $\frac{9}{100}$ inch. How much smaller is a space on the vernier than a space on the ruler? The ruler space is $\frac{1}{10}$ inch, or $\frac{10}{100}$ inch, and the vernier space is $\frac{9}{100}$ inch. The vernier space is smaller by the difference between these two numbers, as follows:

$$\frac{10}{100} - \frac{9}{100} = \frac{1}{100}$$

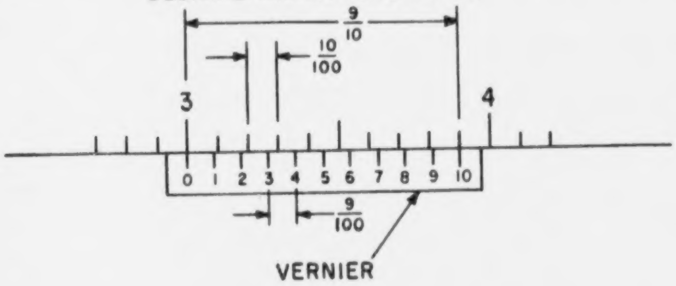


Figure 6-3.—Vernier scale.

Each vernier space is $\frac{1}{100}$ inch smaller than a ruler space.

As an example of the use of the vernier scale, suppose that we are measuring the steel bar shown in figure 6-4. The end of the bar almost reaches the 3-inch mark on the ruler, and we estimate that it is about halfway between 2.9 inches and 3.0 inches. The vernier marks help us to decide whether the exact measurement is 2.94 inches, 2.95 inches, or 2.96 inches.

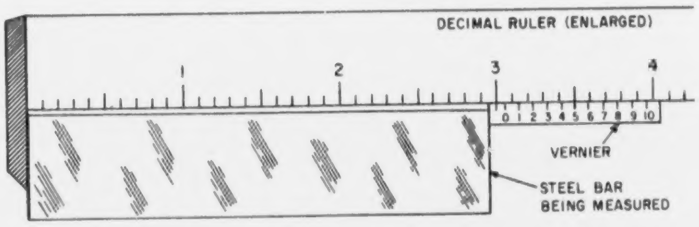


Figure 6-4.—Measuring with a vernier.

The 0 on the vernier scale is spaced the distance of exactly one ruler mark (in this case, one tenth of an inch) from the left hand end of the vernier. Therefore the 0 is at a position between ruler marks which is comparable to the position of the end of the bar. In other words, the 0 on the vernier is about halfway between two adjacent marks on the ruler, just as the end of the bar is about halfway between two adjacent marks. The 1 on the vernier scale is a little closer to alinement with an adjacent ruler mark; in fact, it is one hundredth of an inch closer to alinement than the 0. This is because each space on the vernier is one hundredth of an inch shorter than each space on the ruler.

Each successive mark on the vernier scale is one hundredth of an inch closer to alinement than the preceding mark, until finally alinement is achieved at the 5 mark. This means that the 0 on the vernier must be five hundredths of an inch from the nearest ruler mark, since five increments, each one hundredth of an inch in size, were used before a mark was found in alinement.

We conclude that the end of the bar is five hundredths of an inch from the 2.9 mark on the ruler, since its position between marks is exactly comparable to that of the 0 on the vernier scale. Thus the value of our measurement is 2.95 inches.

The foregoing example could be followed through for any distance between markings. Suppose the 0 mark fell seven tenths of the distance between ruler markings. It would take seven vernier markings, a loss of one-hundredth of an inch each time, to bring the marks in line at 7 on the vernier.

The vernier principle may be used to get fine linear readings, angular readings, etc. The principle is always the same. The vernier has one more marking than the number of markings on an equal space of the conventional scale of the measuring instrument. For example, the vernier caliper (fig. 6-5) has 25 markings on the vernier for 24 on the caliper scale. The caliper is marked off to read to fortieths (0.025) of an inch, and the vernier extends the accuracy to a thousandth of an inch.

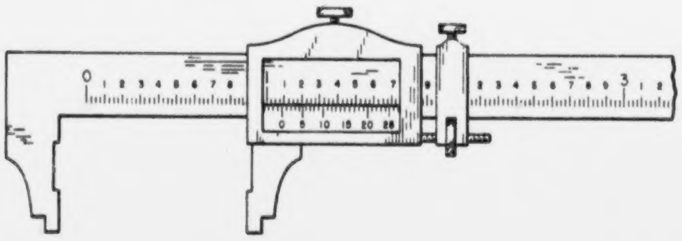


Figure 6-5.—A vernier caliper.

Vernier Micrometer

By adding a vernier to the micrometer, it is possible to read accurately to one ten-thousandth of an inch. The vernier markings are on the sleeve of the micrometer and are parallel to the thimble markings. There are 10 divisions on the vernier that occupy the same space as 9 divisions on the thimble. Since a thimble space is one thousandth of an inch, a vernier space is $\frac{1}{10}$ of $\frac{9}{1000}$ inch, or $\frac{9}{10000}$ inch. It is $\frac{1}{10000}$ inch less than a thimble space. Thus, as in the preceding explanation of verniers, it is possible to read the nearest ten-thousandth of an inch by reading the vernier digit whose marking coincides with a thimble marking.

In figure 6-6 (A), the last major division showing fully on the sleeve index is 3. The third minor division is the last mark clearly showing (0.075). The thimble division nearest and below the index is the 8 (0.008). The vernier marking that matches a thimble marking is the fourth (0.0004). Adding them all together, we have,

$$\begin{array}{r} 0.3000 \\ 0.0750 \\ 0.0080 \\ \underline{0.0004} \\ 0.3834 \end{array}$$

The reading is 0.3834 inch. With practice these readings can be made directly from the micrometer, without writing the partial readings.

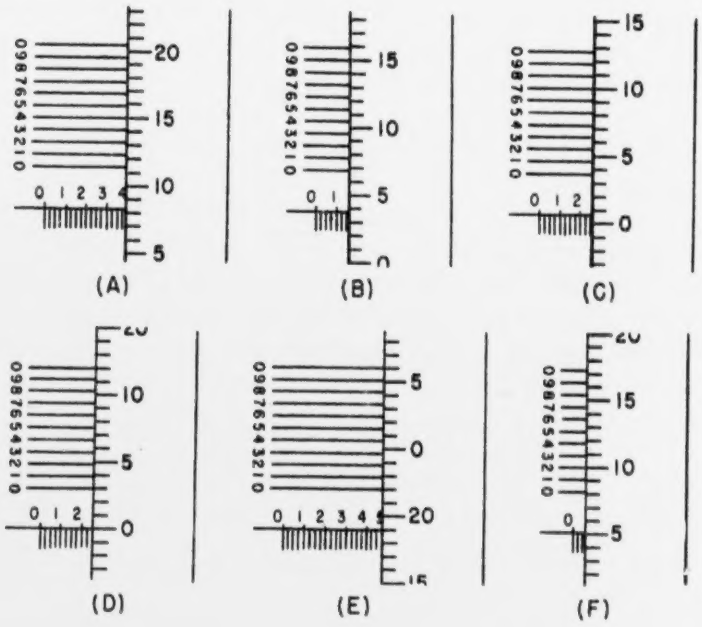


Figure 6-6.—Vernier micrometer settings.

Practice problems:

1. Read the micrometer settings in figure 6-6.

Answers:

1. (A) See the foregoing example.

(B)	0.1539	(E)	0.4690
(C)	0.2507	(F)	0.0552
(D)	0.2500		

CHAPTER 7

EXPONENTS AND RADICALS

The operation of raising a number to a power is a special case of multiplication in which the factors are all equal. In examples such as $4^2 = 4 \times 4 = 16$ and $5^3 = 5 \times 5 \times 5 = 125$, the number 16 is the second power of 4 and the number 125 is the third power of 5. The expression 5^3 means that three 5's are to be multiplied together. Similarly, 4^2 means 4×4. The first power of any number is the number itself. The power is the number of times the number itself is to be taken as a factor.

The process of finding a root is the inverse of raising a number to a power. A root is a special factor of a number, such as 4 in the expression $4^2 = 16$. When a number is taken as a factor two times, as in the expression $4 \times 4 = 16$, it is called a square root. Thus, 4 is a square root of 16. By the same reasoning, 2 is a cube root of 8, since $2 \times 2 \times 2$ is equal to 8. This relationship is usually written as $2^3 = 8$.

POWERS AND ROOTS

A power of a number is indicated by an EXPONENT, which is a number in small print placed to the right and toward the top of the number. Thus, in $4^3 = 64$, the number 3 is the EXPONENT of the number 4. The exponent 3 indicates that the number 4, called the BASE, is to be raised to its third power. The expression is read "4 to the third power (or 4 cubed) equals 64." Similarly, $5^2 = 25$ is read "5 to the second power (or 5 squared) equals 25." Higher powers are read according to the degree indicated; for example, "fourth power," "fifth power," etc.

When an exponent occurs, it must always be written unless its value is 1. The exponent 1 usually is not written, but is understood. For example, the number 5 is actually 5^1. When we work with exponents, it is important to remember that any number that has no written exponent really has an exponent equal to 1.

A root of a number can be indicated by placing a radical sign, $\sqrt{\ }$, over the number and showing the root by placing a small number within the notch of the radical sign. Thus, $\sqrt[3]{64}$ indicates the cube root of 64, and $\sqrt[5]{32}$ indicates the fifth root of 32. The number that indicates the root is called the INDEX of the root. In the case of the square root, the index, 2, usually is not shown. When a radical has no index, the square root is understood to be the one desired. For example, $\sqrt{36}$ indicates the square root of 36. The line above the number whose root is to be found is a symbol of grouping called the vinculum. When the radical symbol is used, a vinculum, long enough to extend over the entire expression whose root is to be found, should be attached.

Practice problems. Raise to the indicated power or find the root indicated.

1. 2^3 2. 6^2 3. 4^3 4. 25^3
5. $\sqrt{16}$ 6. $\sqrt[3]{8}$ 7. $\sqrt[3]{125}$ 8. $\sqrt[5]{32}$

Answers:

1. 8 2. 36 3. 64 4. 15,625
5. 4 6. 2 7. 5 8. 2

NEGATIVE INTEGERS

Raising to a power is multiplication in which all the numbers being multiplied together are equal. The sign of the product is determined, as in ordinary multiplication, by the number of minus signs. The number of minus signs is odd or even, depending on whether the exponent of the power is odd or even. For example, in the problem

$$(-2)^3 = (-2)(-2)(-2) = -8$$

there are three minus signs. The result is negative. In

$$(-2)^6 = 64$$

there are six minus signs. The result is positive.

Thus, when the exponent of a negative number is odd, the power is negative; when the exponent is even, the power is positive.

As other examples, consider the following:

$$(-3)^4 = 81$$

$$\left(-\frac{2}{5}\right)^3 = -\frac{8}{125}$$

$$(-2)^8 = 256$$

$$(-1)^5 = -1$$

Positive and negative numbers belong to the class called REAL NUMBERS. The square of a real number is positive. For example, $(-7)^2 = 49$ and $7^2 = 49$. The expression $(-7)^2$ is read "minus seven squared." Note that either seven squared or minus seven squared gives us +49. We cannot obtain -49 or any other negative number by squaring any real number, positive or negative.

Since there is no real number whose square is a negative number, it is sometimes said that the square root of a negative number does not exist. However, an expression under a square root sign may take on negative values. While the square root of a negative number cannot actually be found, it can be indicated.

The indicated square root of a negative number is called an IMAGINARY NUMBER. The number $\sqrt{-7}$, for example, is said to be imaginary. It is read "square root of minus seven." Imaginary numbers are discussed in chapter 15 of this course.

FRACTIONS

We recall that the exponent of a number tells the number of times that the number is to be taken as a factor. A fraction is raised to a power by raising the numerator and the denominator separately to the power indicated. The expression $\left(\frac{3}{7}\right)^2$ means $\frac{3}{7}$ is used twice as a factor. Thus,

$$\left(\frac{3}{7}\right)^2 = \frac{3}{7} \times \frac{3}{7} = \frac{3^2}{7^2}$$

$$= \frac{9}{49}$$

Similarly,

$$\left(-\frac{1}{5}\right)^2 = \frac{1}{25}$$

Since a minus sign can occupy any one of three locations in a fraction, notice that evaluating $\left(-\frac{1}{5}\right)^2$ is equivalent to

$$(-1)^2 \left(\frac{1}{5}\right)^2 \text{ or } \frac{(-1)^2}{5^2} \text{ or } \frac{1^2}{(-5)^2}$$

The process of taking a root of a number is the inverse of the process of raising the number to a power, and the method of taking the root of a fraction is similar. We may simply take the root of each term separately and write the result as a fraction. Consider the following examples:

1. $\sqrt{\frac{36}{49}} = \frac{\sqrt{36}}{\sqrt{49}} = \frac{6}{7}$
2. $\sqrt[3]{\frac{8}{125}} = \frac{\sqrt[3]{8}}{\sqrt[3]{125}} = \frac{2}{5}$

Practice problems. Find the values for the indicated operations:

1. $\left(\frac{1}{3}\right)^2$ 2. $\left(\frac{3}{4}\right)^2$ 3. $\left(\frac{6}{5}\right)^2$ 4. $\left(\frac{2}{3}\right)^3$

5. $\sqrt{\frac{16}{36}}$ 6. $\sqrt{\frac{16}{25}}$ 7. $\sqrt[3]{\frac{8}{27}}$ 8. $\sqrt{\frac{9}{49}}$

Answers:

1. 1/9 2. 9/16 3. 36/25 4. 8/27

5. 4/6 6. 4/5 7. 2/3 8. 3/7

DECIMALS

When a decimal is raised to a power, the number of decimal places in the result is equal to the number of places in the decimal multiplied by the exponent. For example, consider $(0.12)^3$. There are two decimal places in 0.12 and 3 is the exponent. Therefore, the number of places in the power will be 3(2) = 6. The result is as follows:

$$(0.12)^3 = 0.001728$$

The truth of this rule is evident when we recall the rule for multiplying decimals. Part of the rule states: Mark off as many decimal places in the product as there are decimal places in the factors together. If we carry out the multiplication, (0.12) x (0.12) x (0.12), it is obvious that there are six decimal places in the three factors together. The rule can be shown for any decimal raised to any power by simply carrying out the multiplication indicated by the exponent.

Consider these examples:

$$(1.4)^2 = 1.96$$
$$(0.12)^2 = 0.0144$$
$$(0.4)^3 = 0.064$$
$$(0.02)^2 = 0.0004$$
$$(0.2)^2 = 0.04$$

Finding a root of a number is the inverse of raising a number to a power. To determine the number of decimal places in the root of a perfect power, we divide the number of decimal places in the radicand by the index of the root. Notice that this is just the opposite of what was done in raising a number to a power.

Consider $\sqrt{0.0625}$. The square root of 625 is 25. There are four decimal places in the radicand, 0.0625, and the index of the root is 2. Therefore, $4 \div 2 = 2$ is the number of decimal places in the root. We have

$$\sqrt{0.0625} = 0.25$$

Similarly,

$$\sqrt{1.69} = 1.3$$
$$\sqrt[3]{0.027} = 0.3$$
$$\sqrt[3]{1.728} = 1.2$$
$$\sqrt[4]{0.0001} = 0.1$$

LAWS OF EXPONENTS

All of the laws of exponents may be developed directly from the definition of exponents. Separate laws are stated for the following five cases:

1. Multiplication.
2. Division.
3. Power of a power.
4. Power of a product.
5. Power of quotient.

MULTIPLICATION

To illustrate the law of multiplication, we examine the following problem:

$$4^3 \times 4^2 = ?$$

Recalling that 4^3 means 4 x 4 x 4 and 4^2 means 4 x 4, we see that 4 is used as a factor five times. Therefore $4^3 \times 4^2$ is the same as 4^5. This result could be written as follows:

$$4^3 \times 4^2 = 4 \times 4 \times 4 \times 4 \times 4$$
$$= 4^5$$

Notice that three of the five 4's came from the expression 4^3, and the other two 4's came from the expression 4^2. Thus we may rewrite the problem as follows:

$$4^3 \times 4^2 = 4^{(3+2)}$$
$$= 4^5$$

The law of exponents for multiplication may be stated as follows: To multiply two or more powers having the same base, add the exponents and raise the common base to the sum of the exponents. This law is further illustrated by the following examples:

$$2^3 \times 2^4 = 2^7$$
$$3 \times 3^2 = 3^3$$
$$15^4 \times 15^2 = 15^6$$
$$10^2 \times 10^{0.5} = 10^{2.5}$$

Common Errors

It is important to realize that the base must be the same for each factor, in order to apply the laws of exponents. For example, $2^3 \times 3^2$ is neither 2^5 nor 3^5. There is no way to apply the law of exponents to a problem of this kind. Another common mistake is to multiply the bases together. For example, this kind of error in the foregoing problem would imply that $2^3 \times 3^2$ is equivalent to 6^5, or 7776. The error of this may be proved as follows:

$$2^3 \times 3^2 = 8 \times 9$$
$$= 72$$

DIVISION

The law of exponents for division may be developed from the following example:

$$6^7 \div 6^5 = \frac{\cancel{6} \times \cancel{6} \times \cancel{6} \times \cancel{6} \times \cancel{6} \times 6 \times 6}{\cancel{6} \times \cancel{6} \times \cancel{6} \times \cancel{6} \times \cancel{6}}$$
$$= 6^2$$

Cancellation of the five 6's in the divisor with five of the 6's in the dividend leaves only two 6's, the product of which is 6^2.

This result can be reached directly by noting that 6^2 is equivalent to $6^{(7-5)}$. In other words, we have the following:

$$\begin{aligned} 6^7 \div 6^5 &= 6^{(7-5)} \\ &= 6^2 \end{aligned}$$

Therefore the law of exponents for division is as follows: To divide one power into another having the same base, subtract the exponent of the divisor from the exponent of the dividend. Use the number resulting from this subtraction as the exponent of the base in the quotient.

Use of this rule sometimes produces a negative exponent or an exponent whose value is 0. These two special types of exponents are discussed later in this chapter.

POWER OF A POWER

Consider the example $(3^2)^4$. Remembering that an exponent shows the number of times the base is to be taken as a factor and noting in this case that 3^2 is considered the base, we have

$$(3^2)^4 = 3^2 \cdot 3^2 \cdot 3^2 \cdot 3^2$$

Also in multiplication we add exponents. Thus,

$$3^2 \cdot 3^2 \cdot 3^2 \cdot 3^2 = 3^{(2+2+2+2)} = 3^8$$

Therefore,

$$\begin{aligned} (3^2)^4 &= 3^{(4 \times 2)} \\ &= 3^8 \end{aligned}$$

The laws of exponents for the power of a power may be stated as follows: To find the power of a power, multiply the exponents. It should be noted that this case is the only one in which multiplication of exponents is performed.

POWER OF A PRODUCT

Consider the example $(3 \cdot 2 \cdot 5)^3$. We know that

$$(3 \cdot 2 \cdot 5)^3 = (3 \cdot 2 \cdot 5)(3 \cdot 2 \cdot 5)(3 \cdot 2 \cdot 5)$$

Thus 3, 2, and 5 appear three times each as factors, and we can show this with exponents as 3^3, 2^3, and 5^3. Therefore,

$$(3 \cdot 2 \cdot 5)^3 = 3^3 \cdot 2^3 \cdot 5^3$$

The law of exponents for the power of a product is as follows: The power of a product is equal to the product obtained when each of the original factors is raised to the indicated power and the resulting powers are multiplied together.

POWER OF A QUOTIENT

The law of exponents for a power of an indicated quotient may be developed from the following example:

$$\begin{aligned} \left(\frac{2}{3}\right)^3 &= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} \\ &= \frac{2 \cdot 2 \cdot 2}{3 \cdot 3 \cdot 3} \\ &= \frac{2^3}{3^3} \end{aligned}$$

Therefore,

$$\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3}$$

The law is stated as follows: The power of a quotient is equal to the quotient obtained when the dividend and divisor are each raised to the indicated power separately, before the division is performed.

Practice problems. Raise each of the following expressions to the indicated power:

1. $(3^2 \cdot 2^3)^2$
2. $3^5 \div 3^2$
3. $\left(\frac{3 \cdot 2}{5 \cdot 6}\right)^3$
4. $(-3^2)^3$
5. $\frac{5^3}{5}$
6. $(3 \cdot 2 \cdot 7)^2$

Answers:

1. $3^4 \times 2^6 = 5{,}184$
2. 27
3. $\frac{1}{125}$
4. $[(-3)^2]^3 = 729$
5. 25
6. $9 \cdot 4 \cdot 49 = 1{,}764$

SPECIAL EXPONENTS

Thus far in this discussion of exponents, the mphasis has been on exponents which are posiive integers. There are two types of exponents vhich are not positive integers, and two which ire treated as special cases even though they nay be considered as positive integers.

ZERO AS AN EXPONENT

Zero occurs as an exponent in the answer to a problem such as $4^3 \div 4^3$. The law of exponents for division states that the exponents are to be subtracted. This is illustrated as follows:

$$\frac{4^3}{4^3} = 4^{(3-3)} = 4^0$$

Another way of expressing the result of dividing 4^3 by 4^3 is to use the fundamental axiom which states that any number divided by itself is 1. In order for the laws of exponents to hold true in all cases, this must also be true when any number raised to a power is divided by itself. Thus, $4^3/4^3$ must equal 1.

Since $4^3/4^3$ has been shown to be equal to both 4^0 and 1, we are forced to the conclusion that $4^0 = 1$.

By the same reasoning,

$$\frac{5}{5} = 5^{1-1} = 5^0$$

Also,

$$\frac{5}{5} = 1$$

Therefore,

$$5^0 = 1$$

Thus we see that any number divided by itself results in a 0 exponent and has a value of 1. By definition then, any number (other than zero) raised to the zero power equals 1. This is further illustrated in the following examples:

$$3^0 = 1$$

$$400^0 = 1$$

$$0.02^0 = 1$$

$$\left(\frac{1}{5}\right)^0 = 1$$

$$(\sqrt{3})^0 = 1$$

ONE AS AN EXPONENT

The number 1 arises as an exponent sometimes as a result of division. In the example $\frac{5^3}{5^2}$ we subtract the exponents to get

$$5^{3-2} = 5^1$$

This problem may be worked another way as follows:

$$\frac{5^3}{5^2} = \frac{\cancel{5} \cdot \cancel{5} \cdot 5}{\cancel{5} \cdot \cancel{5}} = 5$$

Therefore,

$$5^1 = 5$$

We conclude that any number raised to the first power is the number itself. The exponent 1 usually is not written but is understood to exist.

NEGATIVE EXPONENTS

If the law of exponents for division is extended to include cases where the exponent of the denominator is larger, negative exponents arise. Thus,

$$\frac{3^2}{3^5} = 3^{2-5} = 3^{-3}$$

Another way of expressing this problem is as follows:

$$\frac{3^2}{3^5} = \frac{\cancel{3} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{3} \cdot 3 \cdot 3 \cdot 3} = \frac{1}{3^3}$$

Therefore,

$$3^{-3} = \frac{1}{3^3}$$

We conclude that a number N with a negative exponent is equivalent to a fraction having the following form: Its numerator is 1; its denominator is N with a positive exponent whose absolute value is the same as the absolute value of the original exponent. In symbols, this rule may be stated as follows:

$$N^{-a} = \frac{1}{N^a}$$

Also,

$$\frac{1}{N^{-a}} = N^a$$

The following examples further illustrate the rule:

$$5^{-1} = \frac{1}{5}$$

$$6^{-2} = \frac{1}{6^2}$$

$$4^{-12} = \frac{1}{4^{12}}$$

$$\frac{1}{3^{-2}} = 3^2$$

Notice that the sign of an exponent may be changed by merely moving the expression which contains the exponent to the other position in the fraction. The sign of the exponent is changed as this move is made. For example,

$$\frac{1}{10^{-2}} = 1 \div \frac{1}{10^2}$$

$$= 1 \text{ x } \frac{10^2}{1}$$

Therefore,

$$\frac{1}{10^{-2}} = \frac{10^2}{1}$$

By using the foregoing relationship, a problem such as $3 \div 5^{-4}$ may be simplified as follows:

$$\frac{3}{5^{-4}} = 3 \text{ x } \frac{1}{5^{-4}}$$

$$= 3 \text{ x } \frac{5^4}{1}$$

$$= 3 \text{ x } 5^4$$

FRACTIONAL EXPONENTS

Fractional exponents obey the same laws as do integral exponents. For example,

$$4^{1/2} \text{ x } 4^{1/2} = 4^{(1/2 + 1/2)}$$

$$= 4^{2/2}$$

$$= 4^1 = 4$$

Another way of expressing this would be

$$4^{1/2} \text{ x } 4^{1/2} = (4^{1/2})^2$$

$$= 4^{(1/2 \times 2)}$$

$$= 4^1 = 4$$

Observe that the number $4^{1/2}$, when squared in the foregoing example, produced the number 4 as an answer. Recalling that a square root of a number N is a number x such that $x^2 = N$, we conclude that $4^{1/2}$ is equivalent to $\sqrt{4}$. Thus we have a definition, as follows: A fractional exponent of the form 1/r indicates a root, the index of which is r. This is further illustrated in the following examples:

$$2^{1/2} = \sqrt{2}$$

$$4^{1/3} = \sqrt[3]{4}$$

$$6^{2/3} = (6^{1/3})^2 = (\sqrt[3]{6})^2$$

Also,

$$6^{2/3} = (6^2)^{1/3} = \sqrt[3]{36}$$

Notice that in an expression such as $8^{2/3}$ we can either find the cube root of 8 first or square 8 first, as shown by the following example:

$$(8^{1/3})^2 = 2^2 = 4 \text{ and } (8^2)^{1/3} = \sqrt[3]{64} = 4$$

All the numbers in the evaluation of $8^{2/3}$ remain small if the cube root is found before raising the number to the second power. This order of operation is particularly desirable in evaluating a number like $64^{5/6}$. If 64 were first raised to the fifth power, a large number would result. It would require a great deal of unnecessary effort to find the sixth root of 64^5. The result is obtained easily, if we write

$$64^{5/6} = (64^{1/6})^5 = 2^5 = 32$$

If an improper fraction occurs in an exponent, such as 7/3 in the expression $2^{7/3}$, it is customary to keep the fraction in that form rather than express it as a mixed number. In fraction form an exponent shows immediately what power is intended and what root is intended. However, $2^{7/3}$ can be expressed in another form and simplified by changing the improper fraction to a mixed number and writing the fractional part in the radical form as follows:

$$2^{7/3} = 2^{2 + 1/3} = 2^2 \cdot 2^{1/3} = 4\sqrt[3]{2}$$

The law of exponents for multiplication may be combined with the rule for fractional exponents to solve problems of the following type:

PROBLEM: Evaluate the expression $4^{2.5}$.

SOLUTION:

$$\begin{aligned} 4^{2.5} &= 4^2 \times 4^{0.5} \\ &= 16 \times 4^{1/2} \\ &= 16 \times 2 \\ &= 32 \end{aligned}$$

Practice problems:

1. Perform the indicated division: $\frac{2}{2^{1/3}}$
2. Find the product: $7^{2/5} \times 7^{1/10} \times 7^{3/10}$
3. Rewrite with a positive exponent and simplify: $9^{-1/2}$
4. Evaluate $100^{3/2}$
5. Evaluate $(8^0)^5$

Answers:

1. $2^{3/3} \div 2^{1/3} = \sqrt[3]{4}$
2. $7^{8/10}$
3. $\frac{1}{9^{1/2}} = \frac{1}{3}$
4. 1,000
5. 1

SCIENTIFIC NOTATION AND POWERS OF 10

Technicians, engineers, and others engaged in scientific work are often required to solve problems involving very large and very small numbers. Problems such as

$$\frac{22{,}684 \times 0.00189}{0.0713 \times 83 \times 7}$$

are not uncommon. Solving such problems by the rules of ordinary arithmetic is laborious and time consuming. Moreover, the tedious arithmetic process lends itself to operational errors. Also there is difficulty in locating the decimal point in the result. These difficulties can be greatly reduced by a knowledge of the powers of 10 and their use.

The laws of exponents form the basis for calculation using powers of 10. The following list includes several decimals and whole numbers expressed as powers of 10:

$$\begin{aligned} 10{,}000 &= 10^4 \\ 1{,}000 &= 10^3 \\ 100 &= 10^2 \\ 10 &= 10^1 \\ 1 &= 10^0 \\ 0.1 &= 10^{-1} \\ 0.01 &= 10^{-2} \\ 0.001 &= 10^{-3} \\ 0.0001 &= 10^{-4} \end{aligned}$$

The concept of scientific notation may be demonstrated as follows:

$$\begin{aligned} 60{,}000 &= 6.0000 \times 10{,}000 \\ &= 6 \times 10^4 \\ 538 &= 5.38 \times 100 \\ &= 5.38 \times 10^2 \end{aligned}$$

Notice that the final expression in each of the foregoing examples involves a number between 1 and 10, multiplied by a power of 10. Furthermore, in each case the exponent of the power of 10 is a number equal to the number of digits between the new position of the decimal point and the original position (understood) of the decimal point.

We apply this reasoning to write any number in scientific notation; that is, as a number between 1 and 10 multiplied by the appropriate power of 10. The appropriate power of 10 is found by the following mechanical steps:

1. Shift the decimal point to standard position, which is the position immediately to the right of the first nonzero digit.
2. Count the number of digits between the new position of the decimal point and its original position. This number indicates the value of the exponent for the power of 10.
3. If the decimal point is shifted to the left, the sign of the exponent of 10 is positive; if the decimal point is shifted to the right, the sign of the exponent is negative.

The validity of this rule, for those cases in which the exponent of 10 is negative, is demonstrated as follows:

$$
\begin{aligned}
0.00657 &= 6.57 \times 0.001 \\
&= 6.57 \times 10^{-3} \\
0.348 &= 3.48 \times 0.1 \\
&= 3.48 \times 10^{-1}
\end{aligned}
$$

Further examples of the use of scientific notation are given as follows:

$$
\begin{aligned}
543{,}000{,}000 &= 5.43 \times 10^{8} \\
186 &= 1.86 \times 10^{2} \\
243.01 &= 2.4301 \times 10^{2} \\
0.0000007 &= 7 \times 10^{-7} \\
0.00023 &= 2.3 \times 10^{-4}
\end{aligned}
$$

Multiplication Using Powers of 10

From the law of exponents for multiplication we recall that to multiply two or more powers to the same base we add their exponents. Thus,

$$10^{4} \times 10^{2} = 10^{6}$$

We see that multiplying powers of 10 together is an application of the general rule. This is demonstrated in the following examples:

1.
$$
\begin{aligned}
10{,}000 \times 100 &= 10^{4} \times 10^{2} \\
&= 10^{4+2} \\
&= 10^{6}
\end{aligned}
$$

2.
$$
\begin{aligned}
0.0000001 \times 0.001 &= 10^{-7} \times 10^{-3} \\
&= 10^{-7+(-3)} \\
&= 10^{-10}
\end{aligned}
$$

3.
$$
\begin{aligned}
10{,}000 \times 0.001 &= 10^{4} \times 10^{-3} \\
&= 10^{4-3} \\
&= 10
\end{aligned}
$$

4.
$$
\begin{aligned}
23{,}000 \times 500 &= \ ? \\
23{,}000 &= 2.3 \times 10^{4} \\
500 &= 5 \times 10^{2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
23{,}000 \times 500 &= 2.3 \times 10^{4} \times 5 \times 10^{2} \\
&= 2.3 \times 5 \times 10^{4} \times 10^{2} \\
&= 11.5 \times 10^{6} \\
&= 11{,}500{,}000
\end{aligned}
$$

5.
$$
\begin{aligned}
62{,}000 \times 0.003 \times 4{,}600 &= \ ? \\
62{,}000 &= 6.2 \times 10^{4} \\
0.0003 &= 3 \times 10^{-4} \\
4{,}600 &= 4.6 \times 10^{3}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
62{,}000 \times 0.0003 \times 4{,}600 &= 6.2 \times 3 \\
&\quad \times 4.6 \times 10^{4} \times 10^{-4} \times 10^{3} \\
&= 85.56 \times 10^{3} \\
&= 85{,}560
\end{aligned}
$$

Practice problems. Multiply, using powers of 10. For the purposes of this exercise, treat all numbers as exact numbers:

1. $10{,}000 \times 0.001 \times 100$
2. $0.000350 \times 5{,}000{,}000 \times 0.0004$
3. $3{,}875 \times 0.000032 \times 3{,}000{,}000$
4. $7{,}000 \times 0.015 \times 1.78$

Answers:

1. 1.0×10^{3}
2. 7.0×10^{-1}
3. 3.72×10^{5}
4. 1.869×10^{2}

Division Using Powers of 10

The rule of exponents for division states that, for powers of the same base, the exponent of the denominator is subtracted from the exponent of the numerator. Thus,

$$
\begin{aligned}
\frac{10^{7}}{10^{3}} &= 10^{7-3} \\
&= 10^{4}
\end{aligned}
$$

It should be remembered that powers may be transferred from numerator to denominator or from denominator to numerator by simply changing the sign of the exponent. The following examples illustrate the use of this rule for powers of 10:

1.
$$
\begin{aligned}
\frac{72{,}000}{0.0012} &= \frac{7.2 \times 10^{4}}{1.2 \times 10^{-3}} \\
&= \frac{7.2}{1.2} \times 10^{4} \times 10^{3} \\
&= 6 \times 10^{7}
\end{aligned}
$$

2. $$\frac{44 \times 10^{-4}}{11 \times 10^{-5}} = \frac{44}{11} \times 10^{-4} \times 10^{5}$$
$$= 4 \times 10$$

Combined Multiplication and Division

Using the rules already shown, multiplication and division involving powers of 10 may be combined. The usual method of solving such problems is to multiply and divide alternately until the problem is completed. For example,

$$\frac{36{,}000 \times 1.1 \times 0.06}{0.012 \times 2{,}200}$$

Rewriting this problem in scientific notation, we have

$$\frac{3.6 \times 10^{4} \times 1.1 \times 6 \times 10^{-2}}{1.2 \times 10^{-2} \times 2.2 \times 10^{3}} = \frac{3.6 \times 1.1 \times 6}{1.2 \times 2.2} \times 10$$
$$= 9 \times 10$$
$$= 90$$

Notice that the elimination of 0's, wherever possible, simplifies the computation and makes it an easy matter to place the decimal point.

SIGNIFICANT DIGITS.—One of the most important advantages of scientific notation is the fact that it simplifies the task of determining the number of significant digits in a number. For example, the fact that the number 0.00045 has two significant digits is sometimes obscured by the presence of the 0's. The confusion can be avoided by writing the number in scientific notation, as follows:

$$0.00045 = 4.5 \times 10^{-4}$$

Practice problems. Express the numbers in the following problems in scientific notation and round off before performing the calculation. In each problem, round off calculation numbers to one more digit than the number of significant digits in the least accurate number; round the answer to the number of significant digits in the least accurate number:

1. $\dfrac{0.000063 \times 50.4 \times 0.007213}{780 \times 0.682 \times 0.018}$
2. $\dfrac{0.015 \times 216 \times 1.78}{72 \times 0.0624 \times 0.0353}$
3. $\dfrac{0.000079 \times 0.00036}{29 \times 10^{-8}}$

Answers:

1. 2.38×10^{-6}
2. 3.64×10
3. 9.8×10^{-2}

Other Applications

The applications of powers of 10 may be broadened to include problems involving reciprocals and powers of products.

RECIPROCALS.—The following example illustrates the use of powers of 10 in the formation of a reciprocal:

$$\frac{1}{250{,}000 \times 300 \times 0.02}$$
$$= \frac{1}{2.5 \times 10^{5} \times 3 \times 10^{2} \times 2 \times 10^{-2}}$$
$$= \frac{10^{-5}}{2.5 \times 3 \times 2}$$
$$= \frac{10^{-5}}{15}$$

Rather than write the numerator as 0.00001, write it as the product of two factors, one of which may be easily divided, as follows:

$$\frac{10^{-5}}{15} = \frac{10^{2} \times 10^{-7}}{15}$$
$$= \frac{100}{15} \times 10^{-7}$$
$$= 6.67 \times 10^{-7}$$
$$= 0.000000667$$

POWER OF A PRODUCT.—The following example illustrates the use of powers of 10 in finding the power of a product:

$$(80{,}000 \times 2 \times 10^{5})^{2} = (8 \times 10^{4} \times 2 \times 10^{5})^{2}$$
$$= 8^{2} \times 2^{2} \times (10^{4+5})^{2}$$
$$= 64 \times 4 \times 10^{18}$$
$$= 256 \times 10^{18}$$
$$= 2.56 \times 10^{20}$$

RADICALS

An expression such as $\sqrt{2}$, $\sqrt[3]{5}$, or $\sqrt{a + b}$ that exhibits a radical sign, is referred to as a

RADICAL. We have already worked with radicals in the form of fractional exponents, but it is also frequently necessary to work with them in the radical form. The word "radical" is derived from the Latin word "radix," which means "root." The word "radix" itself is more often used in modern mathematics to refer to the base of a number system, such as the base 2 in the binary system. However, the word "radical" is retained with its original meaning of "root."

The radical symbol ($\sqrt{\ }$) appears to be a distortion of the initial letter "r" from the word "radix." With long usage, the r gradually lost its significance as a letter and became distorted into the symbol as we use it. The vinculum helps to specify exactly which of the letters and numbers following the radical sign actually belong to the radical expression.

The number under a radical sign is the RADICAND. The index of the root (except in the case of a square root) appears in the trough of the radical sign. The index tells what root of the radicand is intended. For example, in $\sqrt[5]{32}$, the radicand is 32 and the index of the root is 5. The fifth root of 32 is intended. In $\sqrt{50}$, the square root of 50 is intended. When the index is 2, it is not written, but is understood.

If we can find one square root of a number we can always find two of them. Remember $(3)^2$ is 9 and $(-3)^2$ is also 9. Likewise $(4)^2$ and $(-4)^2$ both equal 16 and $(5)^2$ and $(-5)^2$ both equal 25. Conversely, $\sqrt{9}$ is +3 or -3, $\sqrt{16}$ is +4 or -4, and $\sqrt{25}$ is +5 or -5. When we wish to show a number that may be either positive or negative, we may use the symbol ± which is read "plus or minus." Thus ± 3 means "plus or minus 3." Usually when a number is placed under the radical sign, only its positive root is desired and, unless otherwise specified, it is the only root that need be found.

COMBINING RADICALS

A number written in front of another number and intended as a multiplier is called a COEFFICIENT. The expression 5x means 5 times x; ay means a times y; and $7\sqrt{2}$ means 7 times $\sqrt{2}$. In these examples, 5 is the coefficient of x, a is the coefficient of y, and 7 is the coefficient of $\sqrt{2}$.

Radicals having the same index and the same radicand are SIMILAR. Similar radicals may have different coefficients in front of the radical sign. For example, $3\sqrt{2}$, $\sqrt{2}$, and $\frac{1}{5}\sqrt{2}$ are similar radicals. When a coefficient is not written, it is understood to be 1. Thus, the coefficient of $\sqrt{2}$ is 1. The rule for adding radicals is the same as that stated for adding denominate numbers: Add only units of the same kind. For example, we could add $2\sqrt{3}$ and $4\sqrt{3}$ because the "unit" in each of these numbers is the same ($\sqrt{3}$). By the same reasoning, we could not add $2\sqrt{3}$ and $4\sqrt{5}$ because these are not similar radicals.

Addition and Subtraction

When addition or subtraction of similar radicals is indicated, the radicals are combined by adding or subtracting their coefficients and placing the result in front of the radical. Adding $3\sqrt{2}$ and $5\sqrt{2}$ is similar to adding 3 bolts and 5 bolts. The following examples illustrate the addition and subtraction of similar radical expressions:

1. $3\sqrt{2} + 5\sqrt{2} = 8\sqrt{2}$
2. $1/2\ (\sqrt[4]{3}) + 1/3\ (\sqrt[4]{3}) = 5/6\ (\sqrt[4]{3})$
3. $\sqrt{5} - 6\sqrt{5} + 2\sqrt{5} = -3\sqrt{5}$
4. $-5\sqrt[3]{7} - 2\sqrt[3]{7} + 7\sqrt[3]{7} = 0$

Example 4 illustrates a case that is sometimes troublesome. The sum of the coefficients, -5, -2, and 7, is 0. Therefore, the coefficient of the answer would be 0, as follows:

$$0(\sqrt[3]{7}) = 0 \times \sqrt[3]{7}$$

Thus the final answer is 0, since 0 multiplied by any quantity is still 0.

Practice problems. Perform the indicated operations:

1. $4\sqrt{3} - \sqrt{3} + 5\sqrt{3}$
2. $\frac{1}{2}\sqrt{6} + \sqrt{6}$
3. $\sqrt[3]{5} - 6\sqrt[3]{5}$
4. $-2\sqrt{10} - 7\sqrt{10}$

Answers

1. $8\sqrt{3}$
2. $\frac{3}{2}\sqrt{6}$
3. $-5\sqrt[3]{5}$
4. $-9\sqrt{10}$

Multiplication and Division

If a radical is written immediately after another radical, multiplication is intended. Sometimes a dot is placed between the radicals, but not always. Thus, either $\sqrt{7} \cdot \sqrt{11}$ or $\sqrt{7}\,\sqrt{11}$ means multiplication.

When multiplication or division of radicals is indicated, several radicals having the same index can be combined into one radical, if desired. Radicals having the same index are said to be of the SAME ORDER. For example, $\sqrt{2}$ is a radical of the second order. The radicals $\sqrt{2}$ and $\sqrt{5}$ are of the same order.

If radicals are of the same order, the radicands can be multiplied or divided and placed under one radical symbol. For example, $\sqrt{5}$ multiplied by $\sqrt{3}$ is the same as $\sqrt{5 \times 3}$. Also, $\sqrt{6}$ divided by $\sqrt{3}$ is the same as $\sqrt{6 \div 3}$. If coefficients appear before the radicals, they also must be included in the multiplication or division. This is illustrated in the following examples:

1.
$$\begin{aligned} 2\sqrt{2} \cdot 3\sqrt{5} &= 2 \cdot \sqrt{2} \cdot 3 \cdot \sqrt{5} \\ &= 2 \cdot 3\sqrt{2} \cdot \sqrt{5} \\ &= 2 \cdot 3\sqrt{2 \cdot 5} \\ &= 6\sqrt{10} \end{aligned}$$

2.
$$\begin{aligned} \frac{15\sqrt{6}}{3\sqrt{3}} &= \frac{15}{3} \times \sqrt{\frac{6}{3}} \\ &= 5 \times \sqrt{2} \\ &= 5\sqrt{2} \end{aligned}$$

It is important to note that what we have said about multiplication and division does not apply to addition. A typical error is to treat the expression $\sqrt{9 + 4}$ as if it were equivalent to $\sqrt{9} + \sqrt{4}$. These expressions cannot be equivalent, since 3 + 2 is not equivalent to $\sqrt{13}$.

FACTORING RADICALS.—A radical can be split into two or more radicals of the same order if the radicand can be factored. This is illustrated in the following examples:

1. $\sqrt{20} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$
2. $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2}$
   $= \sqrt[3]{27} \cdot \sqrt[3]{2} = 3\sqrt[3]{2}$
3. $\dfrac{\sqrt{20}}{\sqrt{5}} = \dfrac{\sqrt{4} \cdot \sqrt{5}}{\sqrt{5}}$
   $= \sqrt{4} = 2$

SIMPLIFYING RADICALS

Some radicals may be changed to an equivalent form that is easier to use. A radical is in its simplest form when no factor can be removed from the radical, when there is no fraction under the radical sign, and when the index of the root cannot be reduced. A factor can be removed from the radical if it occurs a number of times equal to the index of the root. The following examples illustrate this:

1. $\sqrt{28} = \sqrt{2^2 \cdot 7} = 2\sqrt{7}$
2. $\sqrt[3]{54} = \sqrt[3]{3^3 \cdot 2} = 3(\sqrt[3]{2})$
3. $\sqrt[5]{160} = \sqrt[5]{2^5 \cdot 5} = 2(\sqrt[5]{5})$

Removing a factor that occurs a number of times equal to the index of the root is equivalent to separating a radical into two radicals so that one radicand is a perfect power. The radical sign can be removed from the number that is a perfect square, cube, fourth power, etc. The root taken becomes the coefficient of the remaining radical.

In order to simplify radicals easily, it is convenient to know the squares of whole numbers up to about 25 and a few of the smaller powers of the numbers 2, 3, 4, 5, and 6. Table 7-1 shows some frequently used powers of numbers.

Table 7-1.—Powers of numbers.

$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
$6^2 = 36$
$7^2 = 49$
$8^2 = 64$
$9^2 = 81$
$10^2 = 100$
$11^2 = 121$
$12^2 = 144$
$13^2 = 169$
$14^2 = 196$
$15^2 = 225$
$16^2 = 256$
$17^2 = 289$
$18^2 = 324$
$19^2 = 361$
$20^2 = 400$
$21^2 = 441$
$22^2 = 484$
$23^2 = 529$
$24^2 = 576$
$25^2 = 625$

(A)

Table 7-1.—Powers of numbers—Continued.

$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$
$2^8 = 256$

(B)

$3^1 = 3$
$3^2 = 9$
$3^3 = 27$
$3^4 = 81$
$3^5 = 243$

(C)

$4^1 \div 4$
$4^2 = 16$
$4^3 = 64$
$4^4 = 256$

(D)

$5^1 = 5$
$5^2 = 25$
$5^3 = 125$
$5^4 = 625$

(E)

$6^1 = 6$
$6^2 = 36$
$6^3 = 216$

(F)

Referring to table 7-1 (A), we see that the series of numbers

1, 4, 9, 16, 25, 36, 49, 64, 81, 100

comprises all the perfect squares from 1 to 100 inclusive. If any one of these numbers appears under a square root symbol, the radical sign can be removed immediately. This is illustrated as follows:

$$\sqrt{25} = 5$$

$$\sqrt{81} = 9$$

A radicand such as 75, which has a perfect square (25) as a factor, can be simplified as follows:

$$\begin{aligned}\sqrt{75} &= \sqrt{25 \cdot 3}\\ &= \sqrt{25} \cdot \sqrt{3}\\ &= 5\ \sqrt{3}\end{aligned}$$

This procedure is further illustrated in the following problems:

1. $\sqrt{8} = \sqrt{4 \cdot 2}$
$\quad = \sqrt{4} \cdot \sqrt{2}$
$\quad = 2\ \sqrt{2}$

2. $\sqrt{72} = \sqrt{36 \cdot 2}$
$\quad = \sqrt{36} \cdot \sqrt{2}$
$\quad = 6\ \sqrt{2}$

By reference to the perfect fourth powers in table 7-1, we may simplify a radical such as $\sqrt[4]{405}$. Noting that 405 has the perfect fourth power 81 as a factor, we have the following:

$$\begin{aligned}\sqrt[4]{405} &= \sqrt[4]{81 \cdot 5}\\ &= \sqrt[4]{81} \cdot \sqrt[4]{5}\\ &= 3\ (\sqrt[4]{5})\end{aligned}$$

As was shown with fractional exponents, taking a root is equivalent to dividing the exponent of a power by the index of the root. If a factor of the radicand has an exponent that is not a multiple of the index of the root, the factor may be separated so that one exponent is divisible by the index, as in

$$\sqrt{3^7} = \sqrt{3^6 \cdot 3} = 3^{6/2} \cdot 3^{1/2} = 3^3 \cdot \sqrt{3} = 27\ \sqrt{3}$$

Consider also

$$\begin{aligned}\sqrt{2^3 \cdot 3^7 \cdot 5} &= \sqrt{2^2 \cdot 2 \cdot 3^6 \cdot 3 \cdot 5}\\ &= 2 \cdot 3^3 (\sqrt{2 \cdot 3 \cdot 5})\\ &= 54\ \sqrt{30}\end{aligned}$$

If the radicand is a large number, the perfect powers that are factors are not always obvious. In such a case the radicand can be separated into prime factors. For example,

$$\begin{aligned}\sqrt{8{,}820} &= \sqrt{2^2 \cdot 3^2 \cdot 5 \cdot 7^2}\\ &= 2 \cdot 3 \cdot 7\ \sqrt{5}\\ &= 42\ \sqrt{5}\end{aligned}$$

Practice problems. Simplify the radicals and reduce to lowest terms:

1. $\dfrac{\sqrt{3} \cdot \sqrt{15}}{\sqrt{5}}$

2. $\dfrac{\sqrt[3]{81}}{\sqrt[3]{27}}$

3. $\dfrac{18(\sqrt[3]{30})}{3(\sqrt[3]{10})}$

4. $\dfrac{\sqrt{8{,}820}}{\sqrt{180}}$

Answers:

. 3 3. $6(\sqrt[3]{3})$

. $\sqrt[3]{3}$ 4. 7

ATIONAL AND RRATIONAL NUMBERS

Real and imaginary numbers make up the umber system of algebra. Imaginary numbers re discussed in chapter 15 of this course. eal numbers are either rational or irrational. he word RATIONAL comes from the word 'ratio." A number is rational if it can be exressed as the quotient, or ratio, of two whole umbers. Rational numbers include fractions ike 2/7, whole numbers, and radicals if the adical sign is removable.

Any whole number is rational. Its denomiator is 1. For instance, 8 equals $\frac{8}{1}$, which is he quotient of two integers. A number like $\sqrt{16}$ is rational, since it can be expressed as he quotient of two integers in the form $\frac{4}{1}$. The ollowing are also examples of rational numbers:

1. $\sqrt{\frac{25}{9}}$, which equals $\frac{5}{3}$
2. -6, which equals $\frac{-6}{1}$
3. $5\frac{2}{7}$, which equals $\frac{37}{7}$

Any rational number can be expressed as the uotient of two integers in many ways. For xample,

$$7 = \frac{7}{1} = \frac{14}{2} = \frac{21}{3} \ldots$$

An IRRATIONAL number is a real number hat cannot be expressed as the ratio of two integers. The numbers $\sqrt{3}$, $5\sqrt{2}$, $\sqrt{7}$, $\frac{3}{8}\sqrt{20}$, and $\frac{2}{\sqrt{5}}$ are examples of irrational numbers.

Rationalizing Denominators

Expressions such as $\frac{7}{\sqrt{2}}$ and $\frac{\sqrt{2}}{5\sqrt{3}}$ have irrational numbers in the denominator. If the denominators are changed immediately to decimals, as in

$$\frac{7}{\sqrt{2}} = \frac{7}{1.4142}$$

the process of evaluating a fraction becomes an exercise in long division. Such a fraction can be evaluated quickly by first changing the denominator to a rational number. Converting a fraction with an irrational number in its denominator to an equivalent fraction with a rational number in the denominator is called RATIONALIZING THE DENOMINATOR.

Multiplying a fraction by 1 leaves the value of the fraction unchanged. Since any number divided by itself equals 1, it follows, for example, that

$$\frac{\sqrt{2}}{\sqrt{2}} = 1$$

If the numerator and denominator of $\frac{7}{\sqrt{2}}$ are each multiplied by $\sqrt{2}$, another fraction having the same value is obtained. The result is

$$\frac{7}{\sqrt{2}} = \frac{7}{\sqrt{2}} \quad \frac{\sqrt{2}}{\sqrt{2}} = \frac{7\sqrt{2}}{2}$$

The denominator of the new equivalent fraction is 2, which is rational. The decimal value of the fraction is

$$\frac{7\sqrt{2}}{2} = \frac{7(1.4142)}{2} = 7(0.7071) = 4.9497$$

To rationalize the denominator in $\frac{\sqrt{2}}{5\sqrt{3}}$ we multiply the numerator and denominator by $\sqrt{3}$. We get

$$\frac{\sqrt{2}}{5\sqrt{3}} = \frac{\sqrt{2}}{5\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{5(3)} = \frac{\sqrt{6}}{15} \text{ or } \frac{1}{15}\sqrt{6}$$

Practice problems. Rationalize the denominator in each of the following:

1. $\frac{6}{\sqrt{2}}$
2. $\frac{\sqrt{5}}{\sqrt{3}}$
3. $\frac{2}{\sqrt{6}}$
4. $\frac{6}{\sqrt{y}}$

Answers:

1. $3\sqrt{2}$
2. $\frac{\sqrt{15}}{3}$
3. $\frac{\sqrt{6}}{3}$
4. $\frac{6\sqrt{y}}{y}$

EVALUATING RADICALS

Any radical expression has a decimal equivalent which may be exact if the radicand is a rational number. If the radicand is not rational, the root may be expressed as a decimal approximation, but it can never be exact. A procedure similar to long division may be used for calculating square root and cube root, and higher roots may be calculated by means of methods based on logarithms and higher mathematics. Tables of powers and roots have been calculated for use in those scientific fields in which it is frequently necessary to work with roots.

SQUARE ROOT PROCESS

The arithmetic process for calculation of square root is outlined in the following paragraphs:

1. Begin at the decimal point and mark the number off into groups of two digits each, moving both to the right and to the left from the decimal point. This may leave an odd digit at the right-hand or left-hand end of the number, or both. For example, suppose that the number whose square root we seek is 9025. The number marked off as specified would be as follows:

$$\sqrt{90'25.}$$

2. Find the greatest number whose square is contained in the left-hand group (90). This number is 9, since the square of 9 is 81. Write 9 above the first group. Square this number (9), place its square below the left-hand group, and subtract, as follows:

$$\begin{array}{l} \;\;9 \\ \sqrt{90'25.} \\ \;\;\underline{81} \\ \;\;\;\;9\ 25 \end{array}$$

Bring down the next group (25) and place it beside the 9, as shown. This is the new dividend (925).

3. Multiply the first digit in the root (9) by 20, obtaining 180 as a trial divisor. This trial divisor is contained in the new dividend (925) five times; thus the second digit of the root appears to be 5. However, this number must be added to the trial divisor to obtain a "true divisor." If the true divisor is then too large to use with the second quotient digit, this digit must be reduced by 1. The procedure for step 3 is illustrated as follows:

$$\begin{array}{ll} & \;\;9\;\;\;5. \\ & \sqrt{90'25.} \\ & \;\;\underline{81} \\ \cancel{180} & \;\;\;\;9\ 25 \\ 185 & \;\;\;\;\underline{9\ 25} \\ & \;\;\;\;0\ 00 \end{array}$$

The number 180, resulting from the multiplication of 9 by 20, is written as a trial divisor beside the new dividend (925), as shown. The quotient digit (5) is then recorded and the trial divisor is adjusted, becoming 185. The trial quotient (180) is crossed out.

4. The true divisor (185) is multiplied by the second digit (5) and the product is placed below the new dividend (925). This step is shown in the illustration for step 3. When the product in step 4 is subtracted from the new dividend, the difference is 0; thus, in this example, the root is exact.

5. In some problems, the difference is not 0 after all of the digits of the original number have been used to form new dividends. Such problems may be carried further by adding 0's on the right-hand end of the original number, just as in normal long division. However, in the square root process the 0's must be added and used in groups of 2.

Practice problems. Find the square root of each of the following numbers:

1. 9.61
2. 123.21
3. 0.0025

Answers:

1. 3.1
2. 11.1
3. 0.05

TABLES OF ROOTS

The decimal values of square roots and cube roots of numbers with as many as 3 or 4 digits can be found from tables. The table in appendix I of this course gives the square roots and cube roots of numbers from 1 to 100. Most of the values given in such tables are approximate numbers which have been rounded off.

For example, the fourth column in appendix I shows that $\sqrt{72} = 8.4853$, to 4 decimal places. By shifting the decimal point we can obtain other square roots. A shift of two places in the decimal point in the radicand corresponds to a shift of one place in the same direction in the square root.

The following examples show the effect, as reflected in the square root, of shifting the location of the decimal point in the number whose square root we seek:

$$\sqrt{72} = 8.4853$$
$$\sqrt{0.72} = 0.84853$$
$$\sqrt{0.0072} = 0.084853$$
$$\sqrt{7,200} = 84.853$$

Cube Root

The fifth column in appendix I shows that the cube root of 72 is 4.1602. By shifting the decimal point we immediately have the cube roots of certain other numbers involving the same digits. A shift of three places in the decimal point in the radicand corresponds to a shift of one place in the same direction in the cube root.

Compare the following examples:

$$\sqrt[3]{72} = 4.1602$$
$$\sqrt[3]{0.072} = 0.41602$$
$$\sqrt[3]{72,000} = 41.602$$

Many irrational numbers in their simplified forms involve $\sqrt{2}$ and $\sqrt{3}$. Since these radicals occur often, it is convenient to remember their decimal equivalents as follows:

$$\sqrt{2} = 1.4142 \text{ and } \sqrt{3} = 1.7321$$

Thus any irrational numbers that do not contain any radicals other than $\sqrt{2}$ or $\sqrt{3}$ can be converted to decimal forms quickly without referring to tables.

For example consider

$$\sqrt{72} = 6\sqrt{2} = 6(1.4142) = 8.485$$
$$\sqrt{27} = 3\sqrt{3} = 3(1.7321) = 5.196$$

Keep in mind that the decimal equivalents of $\sqrt{2}$ and $\sqrt{3}$ as used in the foregoing examples are not exact numbers and the results obtained with them are approximate in the fourth decimal place.

CHAPTER 8

LOGARITHMS AND THE SLIDE RULE

Logarithms represent a specialized use of exponents. By means of logarithms, computation with large masses of data can be greatly simplified. For example, when logarithms are used, the process of multiplication is replaced by simple addition and division is replaced by subtraction. Raising to a power by means of logarithms is done in a single multiplication, and extracting a root reduces to simple division.

DEFINITIONS

In the expression $2^3 = 8$, the number 2 is the base (not to be confused with the base of the number system), and 3 is the exponent which must be used with the base to produce the number 8. The exponent 3 is the logarithm of 8 when the base is 2. This relationship is usually stated as follows: The logarithm of 8 to the base 2 is 3. In general, the logarithm of a number N with respect to a given base is the exponent which must be used with the base to produce N. Table 8-1 illustrates this.

Table 8-1.—Logarithms with various bases.

Exponential form	Logarithmic form
$2^3 = 8$	$\log_2 8 = 3$
$4^2 = 16$	$\log_4 16 = 2$
$5^0 = 1$	$\log_5 1 = 0$
$27^{2/3} = 9$	$\log_{27} 9 = 2/3$

Table 8-1 shows that the logarithmic relationship may be expressed equally well in either of two forms; these are the exponential form and the logarithmic form. Observe, in table 8-1, that the base of a logarithmic expression is indicated by placing a subscript just below and to the right of the abbreviation "log." Observe also that the word "logarithm" is abbreviated without using a period.

The equivalency of the logarithmic and exponential forms may be used to restate the fundamental definition of logarithms in its most useful form, as follows:

$$b^x = N \text{ implies that } \log_b N = x$$

In words, this definition is stated as follows: If the base b raised to the x power equals N, then x is the logarithm of the number N to the base b.

One of the many uses of logarithms may be shown by an example in which the base is 2. Table 8-2 shows the powers of 2 from 0 through 20. Suppose that we wish to use logarithms to multiply the numbers 512 and 256, as follows:

From table 8-2, $512 = 2^9$

$256 = 2^8$

Then $512 \times 256 = 2^9 \times 2^8$

$= 2^{17}$

and from the table again $2^{17} = 131072$

It is seen that the problem of multiplication is reduced to the simple addition of the exponents 9 and 8 and finding the corresponding power in the table.

Table 8-2 (A) shows the base 2 in the exponential form with its corresponding powers. The actual computation in logarithmic work does not require that we record the exponential form. All that is required is that we add the appropriate exponents and have available a table in which we can look up the number corresponding to the new exponent after adding. Therefore, table 8-2 (B) is adequate for our purpose. Solving the foregoing example by this table, we have the following:

$$\log_2 512 = 9$$

$$\log_2 256 = 8$$

$$\log_2 \text{ of the product} = 17$$

Therefore, the number we seek is the one in the table whose logarithm is 17. This number is 131,072. In this example, we found the exponents directly, added them since this was a

Table 8-2.—Exponential and logarithmic tables for the base 2.

(A) Powers of 2 from 0 through 20	(B) Logarithms for the base 2 and corresponding powers	
	Log	Number
$2^{0} = 1$	0	1
$2^{1} = 2$	1	2
$2^{2} = 4$	2	4
$2^{3} = 8$	3	8
$2^{4} = 16$	4	16
$2^{5} = 32$	5	32
$2^{6} = 64$	6	64
$2^{7} = 128$	7	128
$2^{8} = 256$	8	256
$2^{9} = 512$	9	512
$2^{10} = 1024$	10	1024
$2^{11} = 2048$	11	2048
$2^{12} = 4096$	12	4096
$2^{13} = 8192$	13	8192
$2^{14} = 16384$	14	16384
$2^{15} = 32768$	15	32768
$2^{16} = 65536$	16	65536
$2^{17} = 131072$	17	131072
$2^{18} = 262144$	18	262144
$2^{19} = 524288$	19	524288
$2^{20} = 1048576$	20	1048576

multiplication problem, and located the corresponding power. This avoided the unnecessary step of writing the base 2 each time.

Practice problems. Use the logarithms in table 8-2 to perform the following multiplication:

1. 64 x 128
2. 1,024 x 256
3. 128 x 4,096
4. 512 x 2,048

Answers:

1. 8,192
2. 262,144
3. 524,288
4. 1,048,576

NATURAL AND COMMON LOGARITHMS

Many natural phenomena, such as rates of growth and decay, are most easily described in terms of logarithmic or exponential formulas. Furthermore, the geometric patterns in which certain seeds grow (for example, sunflower seeds) is a logarithmic spiral. These facts explain the name "natural logarithms." Natural logarithms use the base e, which is an irrational number approximately equal to 2.71828. This system is sometimes called the Napierian system of logarithms, in honor of John Napier, who is credited with the invention of logarithms.

To distinguish natural logarithms from other logarithmic systems the abbreviation, ln, is sometimes used. When ln appears, the base is understood to be e and need not be shown. For example, either $\log_e 45$ or $\ln 45$ signifies the natural logarithm of 45.

COMMON LOGARITHMS

As has been shown in preceding paragraphs, any number may be used as a base for a system of logarithms. The selection of a base is a matter of convenience. Briggs in 1617 found that base 10 possessed many advantages not obtainable in ordinary calculations with other bases. The selection of 10 as a base proved so satisfactory that today it is used almost exclusively for ordinary calculations. Logarithms with 10 as a base are therefore called COMMON LOGARITHMS.

When 10 is used as a base, it is not necessary to indicate it in writing logarithms. For example,

$$\log 100 = 2$$

is understood to mean the same as

$$\log_{10} 100 = 2$$

If the base is other than 10, it must be specified by the use of a subscript to the right and below the abbreviation "log." As noted in the foregoing discussion of natural logarithms, the use of the distinctive abbreviation "ln" eliminates the need for a subscript when the base is e.

It is relatively easy to convert common logarithms to natural logarithms or vice versa, if necessary. It should be noted further that each system has its peculiar advantages, but for most everyday work, the common system is more often used. A simple relation connects the two systems. If the common logarithm of a number can be found, multiplying by 2.3026 gives the natural logarithm of the number. For example,

$$\log 1.60 = 0.2041$$

$$\ln 1.60 = 2.3026 \times 0.2041$$

$$= 0.4700$$

Thus the natural logarithm of 1.60 is 0.4700, correct to four significant digits.

Conversely, multiplying the natural logarithm by 0.4343 gives the common logarithm of a number. As might be expected, the conversion factor 0.4343 is the reciprocal of 2.3026. This is shown as follows:

$$\frac{1}{2.3026} = 0.4343$$

Positive Integral Logarithms

The derivation of positive whole logarithms is readily apparent. For example, we see in table 8-3 (B) that the logarithm of 10 is 1. The number 1 is simply the exponent of the base 10 which yields 10. This is shown in table 8-3 (A) opposite the logarithmic equation. Similarly,

$10^0 = 1$ . . . . log 1 = 0

$10^2 = 100$ . . . . log 100 = 2

$10^3 = 1{,}000$ . . . . . . log 1,000 = 3

$10^4 = 10{,}000$ . . . . . log 10,000 = 4

Table 8-3.—Exponential and corresponding logarithmic notations using base 10.

10^{-4}	$= \frac{1}{10^4}$	= 0.0001	log	0.0001	= -4
10^{-3}	$= \frac{1}{10^3}$	= 0.001	log	0.001	= -3
10^{-2}	$= \frac{1}{10^2}$	= 0.01	log	0.01	= -2
10^{-1}	$= \frac{1}{10}$	= 0.1	log	0.1	= -1
$10^{-1/2}$	$= \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$	= 0.31623	log	0.31623	= -0.5
					= 0.5 -1
10^0		= 1	log	1	= 0
$10^{1/2}$	$= \sqrt{10}$	= 3.1623	log	3.1623	= 0.5
10^1		= 10	log	10	= 1
$10^{3/2}$	$= 10\sqrt{10}$	= 31.623	log	31.623	= 1.5
10^2		= 100	log	100	= 2
$10^{5/2}$	$= 10^2 (\sqrt{10})$	= 316.23	log	316.23	= 2.5
10^3		= 1,000	log	1,000	= 3
$10^{7/2}$	$= 10^3 (\sqrt{10})$	= 3162.3	log	3162.3	= 3.5
10^4		= 10,000	log	10,000	= 4

Positive Fractional Logarithms

Referring to table 8-3, notice that the logarithm of 1 is 0 and the logarithm of 10 is 1. Therefore, the logarithm of a number between 1 and 10 is between 0 and 1. An easy way to verify this is to consider some numbers between 1 and 10 which are powers of 10; the exponent in each case will then be the logarithm we seek. Of course, the only powers of 10 which produce numbers between 1 and 10 are fractional powers.

EXAMPLE: $10^{1/2} = 3.1623$ (approximately)

$10^{0.5} = 3.1623$

Therefore, $\log 3.1623 = 0.5$

Other examples are shown in the table for $10^{3/2}$, $10^{5/2}$, and $10^{7/2}$. Notice that the number that represents $10^{3/2}$, 31.623, logically enough lies between the numbers representing 10^1 and 10^2—that is, between 10 and 100. Notice also that $10^{5/2}$ appears between 10^2 and 10^3, and $10^{7/2}$ lies between 10^3 and 10^4.

Negative Logarithms

Table 8-3 shows that negative powers of 10 may be fitted into the system of logarithms. We recall that 10^{-1} means $\frac{1}{10}$, or the decimal fraction, 0.1. What is the logarithm of 0.1?

SOLUTION: $10^{-1} = 0.1$; $\log 0.1 = -1$

Likewise $10^{-2} = 0.01$; $\log 0.01 = -2$

Negative Fractional Logarithms

Notice in table 8-3 that negative fractional exponents present no new problem in logarithmic notation. For example, $10^{-1/2}$ means $\frac{1}{\sqrt{10}}$.

$$\frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10} = 0.31623$$

What is the logarithm of 0.31623?

SOLUTION:

$$10^{-1/2} = 0.31623; \log 0.31623 = -\frac{1}{2}$$

$$= -0.5$$

Table 8-3 shows logarithms for numbers ranging from 0.0001 to 10,000. Notice that there are only 8 integral logarithms in the entire range. Excluding zero logarithms, the logarithms for all other numbers in the range are fractional or contain a fractional part. By the year 1628, logarithms for all integers from 1 to 100,000 had been computed. Practically all of these logarithms contain a fractional part. It should be remembered that finding the logarithm of a number is nothing more than expressing the number as a power of 10. Table 8-4 shows the numbers 1 through 10 expressed as powers of 10. Most of the exponents which comprise logarithms are found by methods beyond the scope of this text. However, it is not necessary to know the process used to obtain logarithms in order to make use of them.

Table 8-4.—The numbers 1 through 10 expressed as powers of 10.

$1 = 10^0$	$6 = 10^{0.77815}$
$2 = 10^{0.30103}$	$7 = 10^{0.84510}$
$3 = 10^{0.47712}$	$8 = 10^{0.90309}$
$4 = 10^{0.60206}$	$9 = 10^{0.95424}$
$5 = 10^{0.69897}$	$10 = 10^1$

COMPONENTS OF LOGARITHMS

The fractional part of a logarithm is usually written as a decimal. The whole number part of a logarithm and the decimal part have been given separate names because each plays a special part in relation to the number which the logarithm represents. The whole number part of a logarithm is called the CHARACTERISTIC. This part of the logarithm shows the position of the decimal point in the associated number. The decimal part of a logarithm is called the MANTISSA.

For a particular sequence of digits making up a number, the mantissa of a common logarithm is always the same regardless of the position of the decimal point in that number. For example, $\log 5270 = 3.72181$; the mantissa is 0.72181 and the characteristic is 3.

CHARACTERISTIC

The characteristic of a common logarithm shows the position of the decimal point in the associated number. The characteristic for a given number may be determined by inspection. It will be remembered that a common logarithm is simply an exponent of the base 10.

When we write log 360 = 2.55630, we understand this to mean $10^{2.55630} = 360$. We know that the number is 360 and not 36 or 3,600 because the characteristic is 2. We know 10^1 is 10, 10^2 is 100, and 10^3 is 1,000. Therefore, the number whose value is $10^{2.55630}$ must lie between 100 and 1,000 and of course any number in that range has 3 digits.

Suppose the characteristic had been 1: where would the decimal point in the number be placed? Since 10^1 is 10 and 10^2 is 100, any number whose logarithm is between 1 and 2 must lie between 10 and 100 and will have 2 digits. Notice how the position of the decimal point changes with the value of the characteristic in the following examples:

$$\log 36{,}000 = 4.55630$$

$$\log 3{,}600 = 3.55630$$

$$\log 360 = 2.55630$$

$$\log 36 = 1.55630$$

$$\log 3.6 = 0.55630$$

Note that it is only the characteristic that changes when the decimal point is moved. An advantage of using the base 10 is thus revealed: If the characteristic is known, the decimal point may easily be placed. If the number is known, the characteristic may be determined by inspection; that is, by observing the location of the decimal point.

Although an understanding of the relation of the characteristic to the powers of 10 is necessary for thorough comprehension of logarithms, the characteristic may be determined mechanically by application of the following rules:

1. For a number greater than 1, the characteristic is positive and is one less than the number of digits to the left of the decimal point in the number.
2. For a positive number less than 1, the characteristic is negative and has an absolute value one more than the number of zeros between the decimal point and the first nonzero digit of the number.

Table 8-5 contains examples of each type of characteristic.

Practice problems. In problems 1 through 4, write the characteristic of the logarithm for each number. In 5 through 8, place the decimal

Table 8-5.—Positive and negative characteristics.

Number	Power of 10	Digits in number to the left of decimal point	Characteristic
	Between:		
134	10^2 and 10^3	3	2
13.4	10^1 and 10^2	2	1
1.34	10^0 and 10^1	1	0
		Zeros between decimal point and first nonzero digit	
0.134	10^{-1} and 10^0	0	-1
0.0134	10^{-2} and 10^{-1}	1	-2
0.00134	10^{-3} and 10^{-2}	2	-3

point in each number as indicated by the characteristic (c) given for each.

1. 4,321   2. 1.23   3. 0.05   4. 12
5. 123; c = 4   6. 8,210; c = 0
7. 8; c = -1   8. 321; c = -2

Answers:

1. 3   2. 0   3. -2   4. 1
5. 12,300   6. 8.210   7. 0.8   8. 0.0321

Negative Characteristics

When a characteristic is negative, such as -2, we do not carry out the subtraction, since this would involve a negative mantissa. There are several ways of indicating a negative characteristic. Mantissas as presented in the table in the appendix are always positive and the sign of the characteristic is indicated separately. For example, where log 0.023 = $\bar{2}.36173$, the bar over the 2 indicates that only the characteristic is negative—that is, the logarithm is -2 + 0.36173.

Another way to show the negative characteristic is to place it after the mantissa. In this case we write 0.36173-2.

A third method, which is used where possible throughout this chapter, is to add a certain quantity to the characteristic and to subtract the same quantity to the right of the mantissa. In the case of the example, we may write:

$$\begin{array}{ll} \bar{2}.36173 & \\ \underline{10} & \underline{-10} \\ 8.36173 & -10 \end{array}$$

In this way the value of the logarithm remains the same but we now have a positive characteristic as well as a positive mantissa.

MANTISSA

The mantissa is the decimal part of a logarithm. Tables of logarithms usually contain only mantissas since the characteristic can be readily determined as explained previously. Table 8-6 shows the characteristic, mantissa, and logarithm for several positions of the decimal point using the sequence of digits 4, 5, 6. It will be noted that the mantissa remains the same for that particular sequence of digits, regardless of the position of the decimal point.

Table 8-6.—Effect of changes in the location of the decimal point.

Number	Characteristic	Mantissa	Logarithm
45,600	4	0.6590	4.6590
4,560	3	0.6590	3.6590
456	2	0.6590	2.6590
45.6	1	0.6590	1.6590
4.56	0	0.6590	0.6590
0.456	-1	0.6590	0.6590-1
0.0456	-2	0.6590	0.6590-2
0.00456	-3	0.6590	0.6590-3

Appendix I of this training course is a table which includes the logarithms of numbers from 1 to 100. For our present purpose in using this table, we are concerned only with the first and sixth columns.

The first column contains the number and the sixth column contains its logarithm. For example, if it is desired to find the logarithm of 45, we would find the number 45 in the first column, look horizontally across the page to column 6 and read the logarithm, 1.65321. A glance down the logarithm column will reveal that the logarithms increase in value as the numbers increase in value.

It must be noted in this particular table that both the mantissa and the characteristic are given for the number in the first column. This is simply an additional aid, since the characteristic can easily be determined by inspection.

Suppose that we wish to use the table of Appendix I to find the logarithm of a number not shown in the "number" column. By recalling that the mantissa does not change when the decimal point moves, we may be able to determine the desired logarithm. For example, the number 450 does not appear in the number column of the table. However, the number 45 has the same mantissa as 450; the only difference between the two logs is in their characteristics. Thus the logarithm of 450 is 2.65321.

Practice problems. Find the logarithms of the following numbers:

1. 64 2. 98 3. 6400 4. 9.8

Answers:

1. 1.80618 2. 1.99123

3. 3.80618 4. 0.99123

THE SLIDE RULE

In 1620, not long after the invention of logarithms, Edmond Gunter showed how logarithmic calculations could be carried out mechanically. This is done by laying off lengths on a rule, representing the logarithms of numbers, and by combining these lengths in various ways. The idea was developed and with the contributions of Mannheim in 1851 the slide rule came into being as we know it today.

The slide rule is a mechanical device by which we can carry out any arithmetic calculation with the exception of addition and subtraction. The most common operations with the slide rule are multiplication, division, finding the square or cube of a number, and finding the square root or cube root of a number. Also trigonometric operations are frequently performed. The advantage of the slide rule is that it can be used with relative ease to solve complicated problems. One limitation is that it will give results with a maximum of only three accurate significant digits. This is sufficient in most calculations, however, since most physical constants are only correct to two or three significant digits. When greater accuracy is required, other methods must be used.

A simplified diagram of a slide rule is pictured in figure 8-1. The sliding, central part of the rule is called the SLIDE. The movable glass or plastic runner with a hairline imprinted on it is called the INDICATOR. There is a C scale printed on the slide, and a D scale exactly the same as the C scale printed on the BODY or STOCK of the slide rule. The mark that is associated with the primary number 1 on any slide rule scale is called the INDEX. There is an index at the extreme left and at the extreme right on both the C and D scales. There are other scales, each having a particular use. Some of these will be mentioned later.

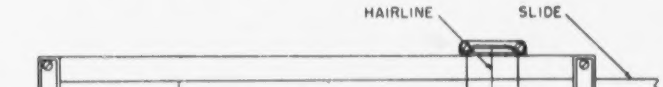


Figure 8-1.—Simplified diagram of a slide rule.

SLIDE RULE THEORY

We have mentioned that the slide rule is based on logarithms. Recall that, to multiply two numbers, we simply add their logarithms. Previously we found these logarithms in tables, but if the logarithms are laid off on scales such as the C and D scale of the slide rule, we can add the lengths, which represent these logarithms. To make such a scale we could mark off mantissas ranging from 0 to 1 on a rule as in figure 8-2. We then find in the tables the logarithms for numbers ranging from 1 to 10 and write the number opposite its corresponding logarithm on the scale.

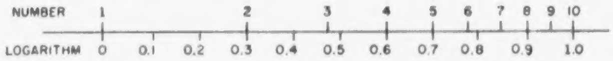


Figure 8-2.—Logarithms and corresponding numbers on a scale.

Table 8-7 lists the numbers 1 through 10 and their corresponding logarithms to three places. These numbers are written opposite their logarithms on the scale shown in figure 8-2. If we have two such scales, exactly alike, arranged so that one of them is free to slide along the other, we can perform the operation of multiplication, for example, by ADDING LENGTHS; that is, by adding logarithms. For example, if we wish to multiply 2 x 3, we find the logarithm of 2 on the stationary scale and move the sliding scale so that its index is over that mark. We then add the logarithm of 3 by finding that logarithm on the sliding scale and by reading below it, on the stationary scale, the logarithm that is the sum of the two.

Since we are not interested in the logarithms themselves, but rather in the numbers they represent, it is possible to remove the logarithmic notation on the scale in figure 8-2, and leave only the logarithmically spaced number scale. The C and D scales of the ordinary slide rule are made up in this manner. Figure 8-3 shows the multiplication of 2 x 3. Although the logarithm scales have been removed, the numbers 2 and 3 in reality signify the logarithms of

Table 8-7.—Numbers and their corresponding logarithms.

Number	Logarithm	Number	Logarithm
1	0.000	6	0.778
2	0.301	7	0.845
3	0.477	8	0.903
4	0.602	9	0.954
5	0.699	10	1.000

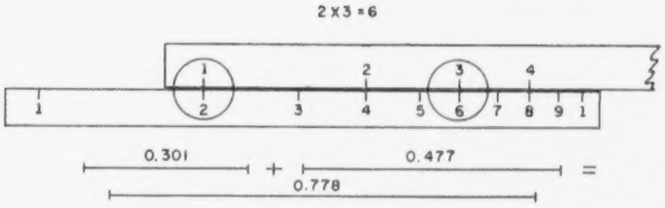


Figure 8-3.—Multiplication by use of the slide rule.

2 and 3, namely, 0.301 and 0.477; the product 6 on the scale really signifies the logarithm of 6, that is, 0.778. Thus, although logarithms are the underlying principle, we are able to work with the numbers directly.

It should be noted that the scale is made up from mantissas only. The characteristic must be determined separately as in the case where tables are used. Since mantissas identify only the digit sequence, the digit 3 on the slide rule represents not only 3 but 30, 300, 0.003, 0.3, and so forth. Thus, the divisions may represent the number multiplied or divided by any power of 10. This is true also for numbers that fall between the divisions. The digit sequence, 1001, could represent 100.1, 1.001, 0.01001, and so forth. The following example shows the use of the same set of mantissas which appear in the foregoing example, but with a different characteristic and, therefore, a different answer:

EXAMPLE: Use logs (positions on the slide rule) to multiply 20 times 30.

SOLUTION:

$$\begin{array}{rl} \log 20 = 1.301 & \text{(2 on the slide rule)} \\ \log 30 = 1.477 & \text{(3 on the slide rule)} \\ \hline \text{log of answer} = 2.778 & \text{(6 on the slide rule)} \end{array}$$

Since the 2 in the log of the answer is merely the indicator of the position of the decimal point in the answer itself, we do not expect to find it on the slide rule scale. As in the foregoing example, we find the digit 6 opposite the multiplier 3. This time, however, the 6 represents 600, because the characteristic of the log represented by 6 in this problem is 2.

READING THE SCALES

Reading a slide rule is no more complicated than reading a yard stick or ruler, if the differences in its markings are understood.

Between the two indices of the C or D scales (the large digit 1 at the extreme left and right of the scales) are divisions numbered 2, 3, 4, 5, 6, 7, 8, and 9. Each length between two consecutive divisions is divided into 10 sections and each section is divided into spaces. (See fig. 8-4.)

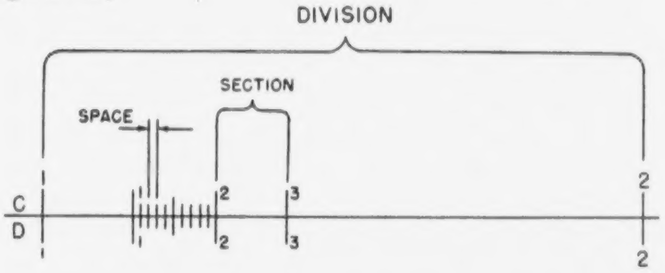


Figure 8-4.—Division, section, and space of a slide rule scale.

Notice that the division between 1 and 2 occupies about one-third of the length of the rule. This is sufficient space in which to write a number for each of the section marks. The sections in the remaining divisions are not numbered, because the space is more limited. Notice also that in the division between 1 and 2, the sections are each divided into 10 spaces. The sections of the divisions from 2 to 4 are subdivided into only 5 spaces, and those from 4 to the right index are subdivided into only 2 spaces. These subdivisions are so arranged because of the limits of space.

Only the sequence of significant digits is read on the slide rule. The position of the decimal point is determined separately. For example, if the hairline of the indicator is in the left-hand position shown in figure 8-5, the significant digits are read as follows:

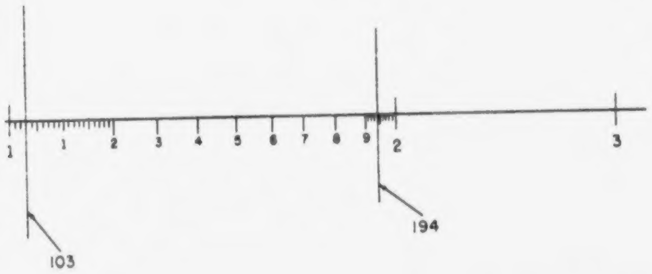


Figure 8-5.—Readings in the first division of a slide rule.

1. Any time the hairline falls in the first division, the first significant digit is 1.
2. Since the hairline lies between the index and the first section mark, we know the number lies between 1.0 and 1.1, or 10 and 11, or 100 and 110, etc. The second significant digit is 0.
3. We next find how far from the index the hairline is located. It lies on the marking for the third space.
4. The three significant digits are 103.

In the second example shown in figure 8-5, the hairline is located in the first division, the ninth section, and on the fourth space mark of that section. Therefore, the significant digits are 194.

Thus, we see that any number falling in the first division of the slide rule will always have 1 as its first significant digit. It can have any number from 0 through 9 as its second digit, and any number from 0 through 9 as its third digit. Sometimes a fourth digit can be roughly approximated in this first division, but the number is really accurate to only three significant digits.

In the second and third divisions, each section is divided into only 5 spaces. (See fig. 8-6.) Thus, each space is equal to 0.2 of the section. Suppose, for example, that the hairline lies on the third space mark after the large 2 indicating the second division. The first significant digit is 2. Since the hairline lies between 2 and the first section mark, the second digit is 0. The hairline lies on the third space mark or 0.6 of the way between the division mark and the first section mark, so the third digit is 6. Thus, the significant digits are 206. Notice that if the hairline lies on a space mark the third digit can be written accurately; otherwise it must be approximated.

From the fourth division to the right index, each section is divided into only two spaces. Thus, if the hairline is in the fourth division and lies on the space mark between the sixth and seventh sections, we would read 465. If the hairline did not fall on a space mark, the third digit would have to be approximated.

OPERATIONS WITH THE SLIDE RULE

There are two parts in solving problems with a slide rule. In the first part the slide rule is used to find the digit sequence of the final result. The second part is concerned with the placing of the decimal point in the result. Let us consider first the digit sequence in multiplication and division.

Multiplication

Multiplication is performed on the C and D scales of the slide rule. The following procedure is used:

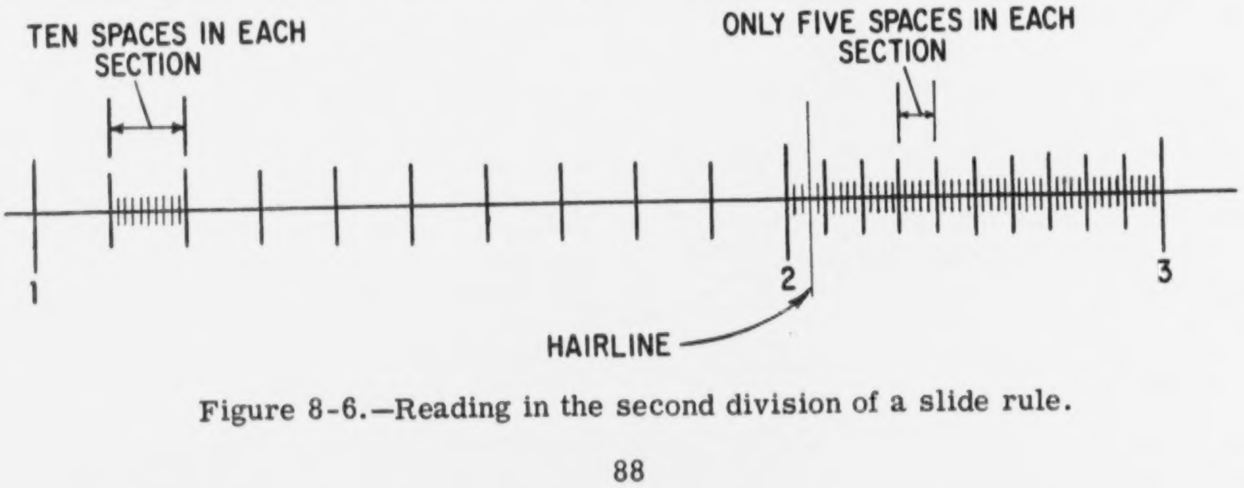


Figure 8-6.—Reading in the second division of a slide rule.

1. Locate one of the factors to be multiplied the D scale, disregarding the decimal point.
2. Place the index of the C scale opposite t number.
3. Locate the other factor on the C scale d move the hairline of the indicator to cover s factor.
4. The product is on the D scale under the rline.

Sometimes in multiplying numbers, such as x 6, the number on the C scale extends to right of the stock and the product cannot be d. In such a case, we simply shift indices. tead of the left-hand index of the C scale, right-hand index is placed opposite the fac- r on the D scale. The rest of the problem mains the same. By shifting indices, we are mply multiplying or dividing by 10, but this ys no part in reading the significant digits. ifting indices affects the characteristic only.

AMPLE: 252 x 3 = 756

1. Place the left index of the C scale over 2.
2. Locate 3 on the C scale and set the hair- e of the indicator over it.
3. Under the hairline on the D scale read product, 756.

AMPLE: 4 x 64 = 256

1. Place the right index of the C scale r 4.
2. Locate 64 on the C scale and set the rline of the indicator over it.
3. Under the hairline on the D scale read product, 256.

Practice problems. Determine the following ducts by slide rule to three significant its:

2.8 x 16 3. 6 x 85
7 x 1.3 4. 2.56 x 3.5

Answers:

44.8 3. 510
9.10 4. 8.96

vision

Division being the inverse of multiplication, process of multiplication is reversed to perform division on a slide rule. We subtract the length representing the logarithm of the divisor from the length representing the logarithm of the dividend to get the logarithm of the quotient.

The procedure is as follows:

1. Locate the dividend on the D scale and place the hairline of the indicator over it.
2. Move the slide until the divisor (on the C scale) lies under the hairline.
3. Read the quotient on the D scale opposite the C scale index.

If the divisor is greater numerically than the dividend, the slide will extend to the left. If the divisor is less, the slide will extend to the right. In either case, the quotient is the number on the D scale that lies opposite the C scale index, falling within the limits of the D scale.

EXAMPLE: 6 ÷ 3 = 2

1. Locate 6 on the D scale and place the hairline of the indicator over it.
2. Move the slide until 3 on the C scale is under the hairline.
3. Opposite the left C scale index, read the quotient, 2, on the D scale.

EXAMPLE: 378 ÷ 63 = 6

1. Locate 378 on the D scale and move the hairline of the indicator over it.
2. Move the slide to the left until 63 on the C scale is under the hairline.
3. Opposite the right-hand index of the C scale, read the quotient, 6, on the D scale.

Practice problems. Determine the following quotients by slide rule.

1. 126 ÷ 3 3. 142 ÷ 71
2. 960 ÷ 15 4. 459 ÷ 17

Answers:

1. 42 3. 2
2. 64 4. 27

PLACING THE DECIMAL POINT

Various methods have been advanced regarding the placement of the decimal point in numbers derived from slide rule computations. Probably the most universal and most easily remembered method is that of approximation.

The method of approximation means simply the rounding off of numbers and the mechanical shifting of decimal points in the numbers of the problem so that the approximate size of the solution and the exact position of the decimal point will be seen from inspection. The slide rule may then be used to derive the correct sequence of significant digits. The method may best be demonstrated by a few examples. Remember, shifting the decimal point in a number one place to the left is the same as dividing by 10. Shifting it one place to the right is the same as multiplying by 10. Every shift must be compensated for in order for the solution to be correct.

EXAMPLE: 0.573×1.45

SOLUTION: No shifting of decimals is necessary here. We see that approximately 0.6 is to be multiplied by approximately 1 1/2. Immediately, we see that the solution is in the neighborhood of 0.9. By slide rule we find that the significant digit sequence of the product is 832. From our approximation we know that the decimal point is to the immediate left of the first significant digit, 8. Thus,

$0.573 \times 1.45 = 0.832$

EXAMPLE: 239×52.3

SOLUTION: For ease in multiplying, we shift the decimal point in 52.3 one place to the left, making it 5.23. To compensate, the decimal point is shifted to the right one place in the other factor. The new position of the decimal point is indicated by the presence of the caret symbol.

$$239.0_{\wedge} \times 5_{\wedge}2.3$$

Our problem is approximately the same as

$$2{,}400 \times 5 = 12{,}000$$

By slide rule the digit sequence is 125. Thus,

$$239 \times 52.3 = 12{,}500$$

EXAMPLE: 0.000134×0.092

SOLUTION:

Shifting decimal points, we have

$$0_{\wedge}00.000134 \times 0.09_{\wedge}2$$

Approximation: $9 \times 0.0000013 = 0.0000117$. By slide rule the digit sequence is 123. From approximation the decimal point is located as follows:

$$0.0000123$$

Thus,

$$0.000134 \times 0.092 = 0.0000123$$

EXAMPLE: $\frac{53.1}{42.4}$

SOLUTION: The decimal points are shifted so that the divisor becomes a number between 1 and 10. The method employed is cancellation.

Shifting decimal points, we have

$$\frac{5_{\wedge}3.1}{4_{\wedge}2.4}$$

Approximation: $\frac{5}{4} = 1.2$

Digit sequence by slide rule:

1255

Placing the decimal point from the approximation:

1.255

Thus,

$$\frac{53.1}{42.4} = 1.255$$

EXAMPLE: $\frac{0.00645}{0.0935}$

SOLUTION:

Shifting decimal points

$$\frac{0.00_{\wedge}645}{0.09_{\wedge}35}$$

Approximation:

$$\frac{0.6}{9} = 0.07$$

Digit sequence by slide rule: 690

Placing the decimal point from the approximation:

$$0.0690$$

Thus,

$$\frac{0.00645}{0.0935} = 0.0690$$

Practice problems. Solve the following problems with the slide rule and use the method of approximation to determine the position of the decimal point:

1. 0.00453×0.1645
2. $53.1 \div 1.255$
3. 0.0362×1.21
4. $67 \div 316$

Answers:

1. 0.000745
2. 42.4
3. 0.0438
4. 0.212

MULTIPLICATION AND DIVISION COMBINED

In problems such as

$$\frac{0.644 \times 330}{161 \times 12}$$

it is generally best to determine the position of the decimal point by means of the method of approximation and to determine the significant digit sequence from the slide rule. Such problems are usually solved by dividing and multiplying alternately throughout the problem. That is, we divide 0.644 by 161, multiply the quotient by 330, and divide that product by 12.

Shifting decimal points, we have

$$\frac{0_{\wedge}0.644 \times 3_{\wedge}30}{1_{\wedge}61 \times 1_{\wedge}2}$$

Since there is a combined shift of three places to the left in the divisor, there must also be a combined shift of three places to the left in the dividend.

$$\text{Approximation: } \frac{0.06 \times \overset{2}{\cancel{3}}}{\cancel{1}.\cancel{5}} = 0.06 \times 2 - 0.12$$

The step-by-step process of determining the significant digit sequence of this problem is as follows:

1. Place the hairline over 644 on the D scale.
2. Draw the slide so that 161 of the C scale lies under the hairline opposite 644.
3. Opposite the C scale index (on the D scale) is the quotient of 644 ÷ 161. This is to be multiplied by 330, but 330 projects beyond the rule so the C scale indices must be shifted.
4. After shifting the indices, find 330 on the C scale and place the hairline over it. Opposite 330 under the hairline on the D scale is the product of $\frac{644}{161} \times 330$.
5. Next, move the C scale until 12 is under the hairline. Opposite the C scale index (on the D scale) is the final quotient. The digit sequence is 110.

The decimal point is then placed according to our approximation: 0.11. Thus,

$$\frac{0.644 \times 330}{161 \times 12} = 0.11$$

Practice problems. Solve the following problems, using a slide rule:

1. $\frac{22 \times 78.5 \times 157}{17 \times 18.3 \times 85}$
2. $\frac{432 \times 9{,}600}{25{,}600 \times 198}$
3. $\frac{2.77 \times 0.064}{0.17 \times 1.97}$

Answers:

1. 10.2
2. 0.817
3. 0.529

SQUARES

Squares of numbers are found by reference to the A scale. The numbers on the A scale are the squares of those on the D Scale. The A scale is really a double scale, each division being one-half as large as the corresponding division on the D scale. The use of a double scale for squaring is based upon the fact that the logarithm of the square of a number is twice as large as the logarithm of the number itself. In other words,

$$\log N^2 = 2 \log N$$

This is reasonable, since

$$\begin{aligned} \log N^2 &= \log (N \times N) \\ &= \log N + \log N \end{aligned}$$

For a numerical example, suppose that we seek to square 2 by means of logarithms.

$$\begin{aligned} \log 2 &= 0.301 \\ \log 2^2 &= 2 \log 2 \\ &= 2 \times 0.301 \\ &= 0.602 \end{aligned}$$

Since each part of the A scale is half as large as the corresponding part of the D scale, the logarithm 0.602 on the A scale will be the same length as the logarithm 0.301 on the D scale. That is, these logarithms will be opposite on the A and D scales. On the A scale as on the D scale, the numbers are written rather than their logarithms. Select several numbers on the D scale, such as 2, 4, 8, 11, and read their squares on the A scale, namely 4, 16, 64, 121.

Notice also that the same relation exists for the B and C scales as for the A and D scales. Of interest, also, is the fact that since the A and B scales are made up as are the C and D scales, they too could be used for multiplying or dividing.

Placing the Decimal Point

Usually the decimal may be placed by the method of approximation. However, close observation will reveal certain facts that eliminate the need for approximations in squaring numbers. Two rules suffice for squaring whole or mixed numbers, as follows:

1. When the square of a number is read on the left half of the A scale, that number will contain twice the number of digits to the left of the decimal point in the original number, less 1.
2. When the square of a number is read on the right half of the A scale, that number will contain twice the number of digits to the left of the decimal point in the original number.

EXAMPLE: Square 2.5.

SOLUTION: Place the hairline over 25 on the D scale. Read the digit sequence, 625, under the hairline in the left half of the A scale.

By rule 1: (2 x number of digits) - 1 = 2(1) - 1 = 1. There is one digit to the left of the decimal point. Thus,

$$(2.5)^2 = 6.25$$

EXAMPLE: Square 6,340.

SOLUTION:

Digit sequence, right half A scale: 402.

By rule 2: 2 x number of digits = 2 x 4 = 8 (digits in answer). Thus,

$$(6{,}340)^2 = 40{,}200{,}000$$

Positive Numbers Less Than One

If positive numbers less than one are to be squared, a slightly different version of the preceding rules must be employed. Count the zeros between the decimal point and the first nonzero digit. Consider this count negative. Then the number of zeros between the decimal point and the first significant digit of the squared number may be found as follows:

1. Left half A scale: Multiply the zeros counted by 2 and subtract 1.
2. Right half A scale: Multiply the zeros counted by 2.

EXAMPLE: Square 0.0045

SOLUTION:

Digit sequence, right half A scale: 2025.

By rule 2: $2(-2) = -4$. (Thus, 4 zeros between the decimal point and the first digit.)

$$(0.0045)^2 = 0.00002025$$

EXAMPLE: Square 0.0215

SOLUTION:

Digit sequence, left half A scale: 462.

By rule 1: $2(-1) - 1 = -3$

$$(0.0215)^2 = 0.000462$$

SQUARE ROOTS

Taking the square root of a number with the slide rule is the inverse process of squaring a number. We find the number on the A scale, set the hairline of the indicator over it, and read the square root on the D scale under the hairline.

Positioning Numbers on the A Scale

Since there are two parts of the A scale exactly alike and the digit sequence could be found on either part, a question arises as to which section to use. Generally, we think of the left half of the rule as being numbered from 1 to 10 and the right half as being numbered from 10 to 100. The numbering continues—left half 100 to 1,000, right half 1,000 to 10,000, and so forth.

A simple process provides a check of the location of the number from which the root is to be taken. For whole or mixed numbers, begin at the decimal point of the number and mark off the digits to the left (including end zeros) in groups of two. This is illustrated in the following two examples:

1. $\sqrt{40{,}300.21}$
   $\sqrt{\text{4'03'00.21}}$
2. $\sqrt{2{,}034.1}$
   $\sqrt{\text{20'34.1}}$

Look at the left-hand group. If it is a 1-digit number, use the left half of the A scale. If it is a 2-digit number, use the right half of the A scale. The number in example 1 is thus located in the left half of the A scale and the number in example 2 is located in the right half.

Numbers Less Than One

For positive numbers less than one, begin at the decimal point and mark off groups of two to the right. This is illustrated as follows:

1. $\sqrt{0.000245}$
   $\sqrt{\text{0.00'02'45}}$
2. $\sqrt{0.00402}$
   $\sqrt{\text{0.00'40'2}}$

Looking from left to right, locate the first group that contains a digit other than zero. If the first figure in this group is zero, locate the number in the left half of the A scale. If the first figure is other than zero, locate the number in the right half of the A scale. Thus,

$\sqrt{\text{0.00'02'45}}$ is located left

and

$\sqrt{\text{0.00'40'2}}$ is located right

Powers of 10

When the square root of 10, 1,000, 100,000, and so forth, is desired, the center index is used. That is, when the number of digits in a power of 10 is even, use the center index.

The slide rule uses only the first three significant digits of a number. Thus, if the rule is used, $\sqrt{23451.6}$ must be considered as $\sqrt{23400.0}$. Likewise, 1.43567 would be considered 1.43000, and so forth. For greater accuracy, other methods must be used.

Practice problems. State which half of the A scale should be used for each of the following:

1. $\sqrt{432}$
2. $\sqrt{0.014}$
3. $\sqrt{241.67}$
4. $\sqrt{0.00045}$
5. $\sqrt{4{,}320}$
6. $\sqrt{0.00301}$
7. $\sqrt{0.0640}$
8. $\sqrt{9.41}$

Answers:

1. Left
2. Left
3. Left
4. Left
5. Right
6. Right
7. Left
8. Left

Placing the Decimal Point

To place the decimal point in the square root of a number, mark off the original number in groups of two as explained previously.

For whole or mixed numbers, the number of groups marked off is the number of digits including end zeros to the left of the decimal point in the root. The following problems illustrate this:

1. $\sqrt{23{,}415}$
   $\sqrt{\text{2'34'15}}$ Three digits to left of decimal point in square root
2. $\sqrt{421{,}562.4}$
   $\sqrt{\text{42'15'62.4}}$ Three digits to left of decimal point in square root
3. $\sqrt{231.321}$
   $\sqrt{\text{2'31.321}}$ Two digits to left of decimal point in square root

For positive numbers less than one, there will be one zero in the square root between the decimal point and the first significant digit for every pair of zeros counted between the decimal point and the first significant digit of the original number. This is illustrated as follows:

1. $\sqrt{0.0004}$

   $\sqrt{0.00'04}$ One zero before first digit in square root

2. $\sqrt{0.00008}$

   $\sqrt{0.00'00'8}$ Two zeros before first digit in square root

3. $\sqrt{0.08'}$ No zeros before first digit in square root

EXAMPLE: $\sqrt{4,521}$

$\sqrt{45'21}$

(Two digits in left-hand group)

Place the hairline over 452 on the right half of the A scale. Read the digit sequence of the root, 672, on the D scale under the hairline. Since there are two groups in the original number, there are two digits to the left of the decimal point in the root. Thus,

$$\sqrt{4,521} = 67.2$$

EXAMPLE: $\sqrt{0.000741}$

$\sqrt{0.00'07'41}$

(First figure is zero in this group)

Place the hairline over 741 on the left half of the A scale. Read the digit sequence of the root, 272, under the hairline on the D scale. Since there is one pair of zeros to the left of the group containing the first digit, there is one zero between the decimal point and the first significant digit of the root. Thus,

$$\sqrt{0.000741} = 0.0272$$

Practice problems. Evaluate each of the following by means of a slide rule:

1. $(17.75)^2$
2. $(0.65)^2$
3. $\sqrt{9.42}$
4. $\sqrt{0.074}$

Answers:

1. 315
2. 0.422
3. 3.07
4. 0.272

CUBES AND CUBE ROOTS

Cubes and cube roots are read on the D and K scales of the slide rule. On the K scale are compressed three complete logarithmic scales in the same space as that of the D scale. Thus, any logarithm on the K scale is three times the logarithm opposite it on the D scale. To cube a number by logarithms, we multiply its logarithm by three. Therefore, the logarithms of cubed numbers will lie on the K scale opposite the numbers on the D scale.

As with the other slide rule scales mentioned, the numbers the logarithms represent, rather than the logarithmic notations, are printed on the rule. In the left-hand third of the K scale, the numbers range from 0 to 10; in the middle third they range from 10 to 100; and in the right-hand third, they range from 100 to 1,000.

To cube a number, find the number on the D scale, place the hairline over it, and read the digit sequence of the cubed number on the K scale under the hairline.

Placing the Decimal Point

The decimal point of a cubed whole or mixed number may be easily placed by application of the following rules:

1. If the cubed number is located in the left third of the K scale, its number of digits to the left of the decimal point is 3 times the number of digits to the left of the decimal point in the original number, less 2.
2. If the cubed number is located in the middle third of the K scale, its number of digits is 3 times the number of digits of the original number, less 1.
3. If the cubed number is located in the right third of the K scale, its number of digits is 3 times the number of digits of the original number.

EXAMPLE: $(1.6)^3$

SOLUTION: Place the hairline over 16 on D scale. Read the digit sequence, 409, on the K scale under the hairline.

Number of digits to left of decimal point in the number 1.6 is 1 and the cubed number is in the left-hand third of the K scale.

$$\begin{aligned} 3 \times (\text{No. of digits})-2 &= (3 \times 1)-2 \\ &= 1 \end{aligned}$$

Therefore,

$$(1.6)^3 = 4.09$$

EXAMPLE: $(4.1)^3$

Digit sequence = 689.

SOLUTION: Number of digits to left of decimal point in the number 4.1 is 1, and the cubed number is in the middle third of the K scale.

$$\begin{aligned} 3 \times (\text{No. of digits})-1 &= (3 \times 1)-1 \\ &= 2 \end{aligned}$$

Therefore,

$$(4.1)^3 = 68.9$$

EXAMPLE: $(52)^3$

SOLUTION: Digit sequence = 141.

Number of digits to left of decimal point in the number 52 is 2, and the cubed number is in the right-hand third of the K scale.

$$\begin{aligned} 3 \times \text{No. of digits} &= 3 \times 2 \\ &= 6 \end{aligned}$$

Therefore,

$$(52)^3 = 141{,}000$$

Positive Numbers Less Than One

If positive numbers less than one are to be cubed, count the zeros between the decimal point and the first nonzero digit. Consider the count negative. Then the number of zeros between the decimal point and the first significant digit of the cubed number may be found as follows:

1. Left third of K scale: Multiply the zeros counted by 3 and subtract 2.
2. Middle third of K scale: Multiply the zeros counted by 3 and subtract 1.
3. Right third of K scale: Multiply the zeros counted by 3.

EXAMPLE: Cube 0.034

SOLUTION: Digit sequence = 393

Zero count of 0.034 = -1, and 393 is in the middle third of the K scale.

$$3 \times (\text{No. of zeros})-1 = (3 \times -1)-1 = -4$$

Therefore,

$$(0.034)^3 = 0.0000393$$

Practice problems. Cube the following numbers using the slide rule.

1. 21   2. 0.7   3. 0.0128   4. 404

Answers:

1. 9260   3. 0.0000021

2. 0.342   4. 66,000,000

Cube Roots

Taking the cube root of a number on the slide rule is the inverse process of cubing a number. To take the cube root of a number, find the number on the K scale, set the hairline over it, and read the cube root on the D scale under the hairline.

POSITIONING NUMBERS ON THE K SCALE.—Since a given number can be located in three positions on the K scale, the question arises as to which third of the K scale to use when locating a number. Generally, the left index, the left middle index, the right middle index, and the right index are considered to be numbered as shown in figure 8-7.

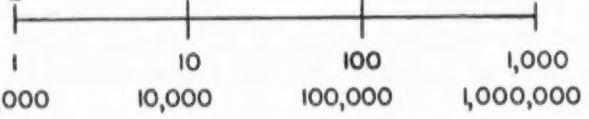


Figure 8-7.—Powers of 10 associated with K-scale indices.

A system similar to that used with square roots may be used to locate the position of a number on the K scale. Groups of three are used rather than groups of two. The grouping for cube root is illustrated as follows:

1. $\sqrt[3]{40{,}531.6}$

   $\sqrt[3]{40'531.6}$

2. $\sqrt[3]{4{,}561.43}$

   $\sqrt[3]{4'561.43}$

3. $\sqrt[3]{0.000043}$

   $\sqrt[3]{0.000'043}$

For whole or mixed numbers the following rules apply:

1. If the left-hand group contains one digit, locate the number in the left third of the K scale.
2. If the left group contains two digits, locate the number in the middle third of the K scale.
3. If the left group contains three digits, locate the number in the right third of the K scale.

The following examples illustrate the foregoing rules:

1. $\sqrt[3]{4'561.43}$

   (One digit)—left third K scale.

2. $\sqrt[3]{40'531.6}$

   (Two digits)—middle third K scale.

3. $\sqrt[3]{453'361}$

   (Three digits)—right third of K scale.

For positive numbers less than one, look from left to right and find the first group that contains a digit other than zero.

1. If the first two figures in this group are zeros, locate the number in the left third of the K scale.
2. If only the first figure in this group is zero, locate the number in the middle third of the K scale.
3. If the first figure of the group is not zero, locate the number in the right third of the K scale.

The following examples illustrate these rules:

1. $\sqrt[3]{0.000'004'53}$

   (Two zeros)—left third K scale.

2. $\sqrt[3]{0.000'050'43}$

   (One zero)—middle third K scale.

3. $\sqrt[3]{0.000'000'430}$

   (No zero)—right third K scale.

PLACING THE DECIMAL POINT.—To place the decimal point in the cube root of a number, we use the system of marking off in groups of three as shown above.

For whole or mixed numbers, there is one digit in the root to the left of the decimal point for every group marked in the original number. Thus,

$$\sqrt[3]{4'531.6}$$

(Two digits in root to left of decimal point.)

For positive numbers less than one, there will be one zero in the root between the decimal point and the first significant digit for every three zeros counted between the decimal point and the first significant digit of the original number. Thus,

$$\sqrt[3]{0.000'000'004}$$

(Two zeros between decimal point and first significant digit of root.)

EXAMPLE: $\sqrt[3]{216000.4}$

$\sqrt[3]{216'000.4}$

(Three digits in left group)

Place the hairline over 216 in the right third of the K scale. Read the digit sequence, 6, under the hairline on the D scale. Since there are two groups in the original number, there are two digits to the left of the decimal point in the root. Thus,

$$\sqrt[3]{216000.4} = 60$$

EXAMPLE: $\sqrt[3]{0.0000451}$

$\sqrt[3]{0.000'045'1}$

(Only first figure is zero in this group)

Place the hairline over 451 in the middle third of the K scale. Read the digit sequence, 357, under the hairline on the D scale. Since there is one group of three zeros, there is one zero between the decimal point and the first significant digit of the root. Thus,

$$\sqrt[3]{0.0000451} = 0.0357$$

POWERS OF 10.—To take the cube root of a power of 10, mark it off as explained in the preceding paragraphs. The number in the left group will then be 1, 10, or 100. We know that the cube root of 10 is a number between 2 and 3. Thus, for the cube root of any number whose left group is 10, use the K scale index which lies between 2 and 3 on the D scale. The cube root of 100 lies between 4 and 5. Therefore, for a number whose left group is 100, use the K scale index between 4 and 5 on the D scale.

Practice problems. Following are some problems and the digit sequence (d. s.) of the roots. Locate the decimal point for each root.

1. $\sqrt[3]{0.000023}$ d. s. 2844
2. $\sqrt[3]{0.051}$ d. s. 371
3. $\sqrt[3]{127}$ d. s. 5026
4. $\sqrt[3]{204{,}000}$ d. s. 589
5. $\sqrt[3]{734{,}000{,}000}$ d. s. 902
6. $\sqrt[3]{4{,}913}$ d. s. 17

Answers:

1. 0.02844
2. 0.371
3. 5.026
4. 58.9
5. 902
6. 17

CHAPTER 9

FUNDAMENTALS OF ALGEBRA

The numbers and operating rules of arithmetic form a part of a very important branch of mathematics called ALGEBRA.

Algebra extends the concepts of arithmetic so that it is possible to generalize the rules for operating with numbers and use these rules in manipulating symbols other than numbers. It does not involve an abrupt change into a distinctly new field, but rather provides a smooth transition into many branches of mathematics with a continuation of knowledge already gained in basic arithmetic.

The idea of expressing quantities in a general way, rather than in the specific terms of arithmetic, is fairly common. A typical example is the formula for the perimeter of a rectangle, $P = 2L + 2W$, in which the letter P represents perimeter, L represents length, and W represents width. It should be understood that $2L = 2(L)$ and $2W = 2(W)$. If the L and the W were numbers, parentheses or some other multiplication sign would be necessary, but the meaning of a term such as 2L is clear without additional signs or symbols.

All formulas are algebraic expressions, although they are not always identified as such. The letters used in algebraic expressions are often referred to as LITERAL NUMBERS (literal implies "letteral").

Another typical use of literal numbers is in the statement of mathematical laws of operation. For example, the commutative, associative, and distributive laws, introduced in chapter 3 with respect to arithmetic, may be restated in general terms by the use of algebraic symbols.

COMMUTATIVE LAWS

The word "commutative" is defined in chapter 3. Remember that the commutative laws refer to those situations in which the factors and terms of an expression are rearranged in a different order.

ADDITION

The algebraic form of the commutative law for addition is as follows:

$$a + b + c = a + c + b = c + b + a$$

In words, this law states that the sum of two or more addends is the same regardless of the order in which the addends are arranged.

The arithmetic example in chapter 3 shows only one specific numerical combination in which the law holds true. In the algebraic example, a, b, and c represent any numbers we choose, thus giving a broad inclusive example of the rule. (Note that once a value is selected for a literal number, that value remains the same wherever the letter appears in that particular example or problem. Thus, if we give a the value of 12, in the example just given, a's value is 12 wherever it appears.)

MULTIPLICATION

The algebraic form of the commutative law for multiplication is as follows:

$$abc = acb = cba$$

In words, this law states that the product of two or more factors is the same regardless of the order in which the factors are arranged.

ASSOCIATIVE LAWS

The associative laws of addition and multiplication refer to the grouping (association) of terms and factors in a mathematical expression.

ADDITION

The algebraic form of the associative law for addition is as follows:

$$a + b + c = (a + b) + c = a + (b + c)$$

In words, this law states that the sum of three or more addends is the same regardless of the manner in which the addends are grouped.

MULTIPLICATION

The algebraic form of the associative law for multiplication is as follows:

$$a \cdot b \cdot c = (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

In words, this law states that the product of three or more factors is the same regardless of the manner in which the factors are grouped.

DISTRIBUTIVE LAW

The distributive law refers to the distribution of factors among the terms of an additive expression. The algebraic form of this law is as follows:

$$a(b + c + d) = ab + ac + ad$$

In words, this law may be stated as follows: If the sum of two or more quantities is multiplied by a third quantity, the product is found by applying the multiplier to each of the original quantities separately and summing the resulting expressions.

ALGEBRAIC SUMS

The word "sum" has been used several times in this discussion, and it is important to realize the full implication where algebra is concerned. Since a literal number may represent either a positive or a negative quantity, a sum of several literal numbers is always understood to be an ALGEBRAIC SUM. That is, it is the sum that results when the algebraic signs of all the addends are taken into consideration.

The following problems illustrate the procedure for finding an algebraic sum:

Let $a = 3$, $b = -2$, and $c = 4$.

$$\begin{aligned} \text{Then } a + b + c &= (3) + (-2) + (4) \\ &= 5 \end{aligned}$$

$$\begin{aligned} \text{Also, } a - b - c &= a + (-b) + (-c) \\ &= 3 + (+2) + (-4) \\ &= 1 \end{aligned}$$

The second problem shows that every expression containing two or more terms to be combined by addition and subtraction may be rewritten as an algebraic sum, all negative signs being considered as belonging to specific terms and all operational signs being positive.

It should be noted, in relation to this subject, that the laws of signs for algebra are the same as those for arithmetic.

ALGEBRAIC EXPRESSIONS

An algebraic expression is made up of the signs and symbols of algebra. These symbols include the Arabic numerals, literal numbers, the signs of operation, and so forth. Such an expression represents one number or one quantity. Thus, just as the sum of 4 and 2 is one quantity, that is, 6, the sum of c and d is one quantity, that is, $c + d$. Likewise $\frac{a}{b}$, $\sqrt{b}$, ab, $a - b$, and so forth, are algebraic expressions each of which represents one quantity or number.

Longer expressions may be formed by combinations of the various signs of operation and the other algebraic symbols, but no matter how complex such expressions are they still represent one number. Thus the algebraic expression $\frac{-a + \sqrt{2a + b}}{6} - c$ is one number

The arithmetic value of any algebraic expression depends on the values assigned to the literal numbers. For example, in the expression $2x^2 - 3ay$, if $x = -3$, $a = 5$, and $y = 1$, then we have the following:

$$\begin{aligned} 2x^2 - 3ay &= 2(-3)^2 - 3(5)(1) \\ &= 2(9) - 15 = 18 - 15 = 3 \end{aligned}$$

Notice that the exponent is an expression such as $2x^2$ applies only to the x. If it is desired to indicate the square of 2x, rather than 2 times the square of x, then parentheses are used and the expression becomes $(2x)^2$.

Practice problems. Evaluate the following algebraic expressions when $a = 4$, $b = 2$, $c = 3$, $x = 7$, and $y = 5$. Remember, the order of operation is multiplication, division, addition, and subtraction.

1. $3x + 7y - c$
2. $xy - 4a^2$
3. $\frac{ax}{b} + y$
4. $c + \frac{ay^2}{b}$

Answers:

1. 53
2. -29
3. 19
4. 53

TERMS AND COEFFICIENTS

The terms of an algebraic expression are the parts of the expression that are connected by plus and minus signs. In the expression $3abx + cy - k$, for example, 3abx, cy, and k are the terms of the expression.

An expression containing only one term, such as 3ab, is called a monomial (mono means one). A binominal contains two terms; for example, $2r + by$. A trinomial consists of three terms. Any expression containing two or more terms may also be called by the general name, polynomial (poly means many). Usually special names are not given to polynomials of more than three times. The expression $x^3 - 3x^2 + 7x + 1$ is a polynomial of four terms. The trinomial $x^2 + 2x + 1$ is an example of a polynomial which has a special name.

Practice problems. Identify each of the following expressions as a monomial, binomial, trinomial, or polynomial. (Some expressions may have two names.)

1. x
2. $3y + a + b$
3. abx
4. $4 + 2b + y + z$
5. $3y^2 + 4$
6. $\frac{2y}{6} + 1$

Answers:

1. Monomial
2. Trinomial (also polynomial)
3. Monomial
4. Polynomial
5. Binomial (also polynomial)
6. Binomial (also polynomial)

In general, a COEFFICIENT of a term is any factor or group of factors of a term by which the remainder of the term is to be multiplied. Thus in the term 2axy, 2ax is the coefficient of y, 2a is the coefficient of xy, and 2 is the coefficient of axy. The word "coefficient" is usually used in reference to that factor which is expressed in Arabic numerals. This factor is sometimes called the NUMERICAL COEFFICIENT. The numerical coefficient is customarily written as the first factor of the term. In 4x, 4 is the numerical coefficient, or simply the coefficient, of x. Likewise, in $24xy^2$, 24 is the coefficient of xy^2 and in $16(a + b)$, 16 is the coefficient of $(a + b)$. When no numerical coefficient is written it is understood to be 1. Thus in the term xy, the coefficient is 1.

COMBINING TERMS

When arithmetic numbers are connected by plus and minus signs, they can always be combined into one number. Thus,

$$5 - 7\frac{1}{2} + 8 = 5\frac{1}{2}$$

Here three numbers are added algebraically (with due regard for sign) to give one number. The terms have been combined into one term.

Terms containing literal numbers can be combined only if their literal parts are the same. Terms containing literal factors in which the same letters are raised to the same power are called like terms. For example, 3y and 2y are like terms since the literal parts are the same. Like terms are added by adding the coefficients of the like parts. Thus, $3y + 2y = 5y$ just as 3 bolts + 2 bolts = 5 bolts. Also $3a^2b$ and a^2b are like; $3a^2b + a^2b = 4a^2b$ and $3a^2b - a^2b = 2a^2b$. The numbers ay and by are like terms with respect to y. Their sum could be indicated in two ways: $ay + by$ or $(a + b)y$. The latter may be explained by comparing the terms to denominate numbers. For instance, a bolts + b bolts = (a + b) bolts.

Like terms are added or subtracted by adding or subtracting the numerical coefficients and placing the result in front of the literal factor, as in the following examples:

$$7x^2 - 5x^2 = (7 - 5)x^2 = 2x^2$$

$$5b^2x - 3ay^2 - 8b^2x + 10ay^2 = -3b^2x + 7ay^2$$

Dissimilar or unlike terms in an algebraic expression cannot be combined when numerical values have not been assigned to the literal factors. For example, $-5x^2 + 3xy - 8y^2$ contains three dissimilar terms. This expression cannot be further simplified by combining terms through addition or subtraction. The expression may be rearranged as $x(3y - 5x) - 8y^2$ or $y(3x - 8y) - 5x^2$, but such a rearrangement is not actually a simplification.

Practice problems. Combine like terms in the following expression:

1. $2a + 4a$
2. $y + y^2 + 2y$
3. $4\frac{ay}{c} - \frac{ay}{c}$
4. $2ay^2 - ay^2$
5. $bx^2 + 2bx^2$
6. $2y + y^2$

Answers:

1. $6a$
2. $y^2 + 3y$
3. $3\frac{ay}{c}$
4. ay^2
5. $3bx^2$
6. $2y + y^2$

SYMBOLS OF GROUPING

Often it is desired to group two or more terms to indicate that they are to be considered and treated as though they were one term even though there may be plus and minus signs between them. The symbols of grouping are parentheses ( ) (which we have already used), brackets [ ], braces { }, and the vinculum ___. The vinculum is sometimes called the "overscore." The fact that $-7 + 2 - 5$ is to be subtracted from 15, for example, could be indicated in any one of the following ways:

$$15 - (-7 + 2 - 5)$$

$$15 - [-7 + 2 - 5]$$

$$15 - \{-7 + 2 - 5\}$$

$$15 - \overline{-7 + 2 - 5}$$

Actually the vinculum is seldom used except in connection with a radical sign, such as in $\sqrt{a + b}$, or in a Boolean algebra expression. Boolean algebra is a specialized kind of symbolic notation which is discussed in Mathematics, Volume 3, NavPers 10073.

Parentheses are the most frequently used symbols of grouping. When several symbols are needed to avoid confusion in grouping, parentheses usually comprise the innermost symbols, followed by brackets, and then by braces as the outermost symbols. This arrangement of grouping symbols is illustrated as follows:

$$2x - \{3y + [-8 - 5y - (x - 4)]\}$$

REMOVING AND INSERTING GROUPING SYMBOLS

Discussed in the following paragraphs are various rules governing the removal and insertion of parentheses, brackets, braces, and the vinculum. Since the rules are the same for all grouping symbols, the discussion in terms of parentheses will serve as a basis for all.

Removing Parentheses

If parentheses are preceded by a minus sign, the entire quantity enclosed must be regarded as a subtrahend. This means that each term of the quantity in parentheses is subtracted from the expression preceding the minus sign. Accordingly, parentheses preceded by a minus sign can be removed, if the signs of all terms within the parentheses are changed.

This may be explained with an arithmetic example. We recall that to subtract one number from another, we change the sign of the subtrahend and proceed as in addition. To subtract -7 from 16, we change the sign of -7 and proceed as in addition, as follows:

$$\begin{aligned} 16 - (-7) &= 16 + 7 \\ &= 23 \end{aligned}$$

It is sometimes easier to see the result of changing signs in the subtrahend if the minus sign preceding the parentheses is regarded as a multiplier. Thus, the thought process in removing parentheses from an expression such as $-(4 - 3 + 2)$ would be as follows: Minus times plus is minus, so the first term of the expression with parentheses removed is -4. (Remember that the 4 in the original expression is understood to be a $+4$, since it has no sign showing.) Minus times minus is plus, so the second term is $+3$. Minus times plus is minus, so the third term is -2. The result is $-4 + 3 - 2$, which reduces to -3.

This same result can be reached just as easily, in an arithmetic expression, by combining the numbers within the parentheses before applying the negative sign which precedes the parentheses. However, in an algebraic expression with no like terms such combination is not possible. The following example shows how the rule for removal of parentheses is applied to algebraic expressions:

$$2a - (-4x + 3by) = 2a + 4x - 3by$$

Parentheses preceded by a plus sign can be removed without any other changes, as the following example shows:

$$2b + (a - b) = 2b + a - b = a + b$$

Many expressions contain more than one set of parentheses, brackets, and other symbols of grouping. In removing symbols of grouping, it is possible to proceed from the outside inward or from the inside outward. For the beginner, it is simpler to start on the inside and work toward the outside, collecting terms and simplifying as one proceeds. In the following example the inner grouping symbols are removed first:

$$\begin{aligned} 2a - [x + (x - 3a) - (9a - 5x)] &= 2a - [x + x - 3a - 9a + 5x] \\ &= 2a - [7x - 12a] \\ &= 2a - 7x + 12a \\ &= 14a - 7x \end{aligned}$$

Enclosing Terms in Parentheses

When it is desired to enclose a group of terms in parentheses, the group of terms remains unchanged if the sign preceding the parentheses is positive. This is illustrated as follows:

$$3x - 2y + 7x - y = (3x - 2y) + (7x - y)$$

Note that this agrees with the rule for removing parentheses preceded by a plus sign.

If terms are enclosed within parentheses preceded by a minus sign, the signs of all the terms enclosed must be changed as in the following example:

$$3x - 2y + 7x - y = 3x - (2y - 7x + y)$$

Practice problems. In problems 1 through 4, remove the symbols of grouping and combine like terms. In problems 5 through 8, enclose the first two terms in parentheses preceded by a plus sign (understood) and the last two in parentheses preceded by a minus sign.

1. $6a - (4a - 3)$
2. $3x + [2x - 4y(6 - 4x)] + 2y - (3 - x + 3y)$
3. $-a + [-a - (2a + 3)] + 3$
4. $(7x - 3ay) - (4a - b) + 16$
5. $4a - 3b - 2c + 4d$
6. $-2 -3x +4y - z$
7. $x + 4y + 3z + 7$
8. $-4 + 2a - 6c + 3d$

Answers:

1. $2a + 3$
2. $6x + 16xy - 25y - 3$
3. $-4a$
4. $7x - 3ay - 4a + b + 16$
5. $(4a - 3b) - (2c - 4d)$
6. $(-2 -3x) - (-4y + z)$
7. $(x + 4y) - (-3z - 7)$
8. $(-4 + 2a) - (6c - 3d)$

EXPONENTS AND RADICALS

Exponents and radicals have the same meaning in algebra as they do in arithmetic. Thus, if n represents any number then $n^2 = n \cdot n$, $n^3 = n \cdot n \cdot n$, etc. By the same reasoning, n^m means that n is to be taken as a factor m times. That is, n^m is equal to $n \cdot n \cdot n \cdot \ldots$, with n appearing m times. The series of dots, called ellipsis (not to be confused with the geometric figure having a similar name, ellipse), represents continuation of the same pattern or the same symbol.

The rules of operation with exponents are also the same in algebra as in arithmetic. For example, $n^2 \cdot n^3 = n^{2+3} = n^5$. Some care is necessary to avoid confusion over an expression such as $3^2 \cdot 3^3$. In this example, $n = 3$ and the product desired is 3^5, not 9^5. In general, $3^a \cdot 3^b = 3^{a+b}$, and a similar result is reached whether the factor which acts as a base for the exponents is a number or a letter. Thus the general form can be expressed as follows:

$$n^a \cdot n^b = n^{a+b}$$

In words, the general rule for multiplication involving exponents is as follows: When multiplying terms whose literal factors are like, the exponents are added. This rule may be applied to problems involving division, if all expressions containing exponents in denominators are rewritten as expressions with negative exponents. For example, the fraction $\frac{x^2y}{xy^2}$ can be rewritten as $(x^2y)(x^{-1}y^{-2})$, which is equal to $(x^{2-1})(y^{1-2})$. This reduces to xy^{-1}, or $\frac{x}{y}$. Notice that the result is the same as it would have been if we had simply subtracted the exponents of literal factors in the denominator from the exponents of the same literal factors in the numerator.

The algebraic rules for radicals also remain the same as those of arithmetic. In arithmetic, $\sqrt{4} = 4^{1/2} = 2$. Likewise, in algebra $\sqrt{a} = a^{1/2}$ and $\sqrt[n]{a} = a^{1/n}$.

MULTIPLYING MONOMIALS

If a monomial such as $3abc$ is to be multiplied by a numerical multiplier, for example 5, the coefficient alone is multiplied, as in the following example:

$$5 \times 3abc = 15abc$$

When the numerical factor is not the initial factor of the expression, as in $x(2a)$, the result of the multiplication is not written as $x2a$. Instead, the numerical factor is interchanged with literal factors by use of the commutative law of multiplication. The literal factors are usually interchanged to place them in alphabetical order, and the final result is as follows:

$$x(2a) = 2ax$$

The rule for multiplication of monomials may be stated as follows: Multiply the numerical coefficients to form the coefficient of the product. Multiply the literal factors, combining exponents in like factors, to form the literal part of the product. The complete process is illustrated in the following example:

$$\begin{aligned}(2ab)(3a^2)(2b^3) &= 12a^{1+2}b^{1+3}\\ &= 12a^3b^4\end{aligned}$$

Practice problems. Perform the indicated operations:

1. $(2x^2)(5x^5)$
2. $(-5ab^2)(2a^2b)$
3. $(-4x^4y)(-3xy^4)$
4. $(2^a)(2^b)$
5. $(-4a^3)^2$
6. $(3a^2b)^2$

Answers:

1. $10x^7$
2. $-10a^3b^3$
3. $12x^5y^5$
4. 2^{a+b}
5. $16a^6$
6. $9a^4b^2$

DIVIDING MONOMIALS

As may be expected, the process of dividing is the inverse of multiplying. Because $3 \times 2a = 6a$, $6a \div 3 = 2a$, or $6a \div 2 = 3a$. Thus, when the divisor is numerical, divide the coefficient of the dividend by the divisor.

When the divisor contains literal parts that are also in the dividend, cancellation may be performed as in arithmetic. For example, $6ab \div 3a$ may be written as follows:

$$\frac{(2)(3a)(b)}{3a}$$

Cancellation of the common literal factor, $3a$, from the numerator and denominator leaves $2b$ as the answer for this division problem.

When the same literal factors appear in both the divisor and the dividend, but with different exponents, cancellation may still be used, as follows:

$$\begin{aligned}\frac{14a^3b^3x}{-21a^2b^5x} &= \frac{(7)(2)a^2ab^3x}{(7)(-3)a^2b^3b^2x}\\ &= \frac{2a}{-3b^2} = -\frac{2a}{3b^2}\end{aligned}$$

This same problem may be solved without thinking in terms of cancellation, by rewriting with negative exponents as follows:

$$\begin{aligned}\frac{14a^3b^3x}{-21a^2b^5x} &= \frac{2a^{3-2}b^{3-5}x^{1-1}}{-3}\\ &= \frac{2ab^{-2}}{-3} = \frac{2a}{-3b^2}\\ &= -\frac{2a}{3b^2}\end{aligned}$$

Practice problems. Perform the indicated operations:

1. $\frac{x^5}{x^6}$
2. $\frac{a^9b^4}{a^6b^3}$
3. $\frac{a^2bc^2}{abc}$
4. $\frac{a^2b}{ab^2}$
5. $\sqrt{16x^4y^6}$
6. $\sqrt{x^{4a}y^{2a}}$
7. $\frac{5a^4b}{10a^2b^3}$
8. $\frac{10x^2y^3z^4}{-5xy^2z^3}$
9. $\sqrt{100a^8b^4}$
10. $\sqrt{a^6b^{6n}}$

Answers:

1. x^{-1}
2. a^3b
3. ac
4. $\frac{a}{b}$
5. $\pm 4x^2y^3$
6. $\pm x^{2a}y^a$
7. $\frac{a^2}{2b^2}$
8. $-2xyz$
9. $\pm 10a^4b^2$
10. $\pm a^3b^{3n}$

OPERATIONS WITH POLYNOMIALS

Adding and subtracting polynomials is simply the adding and subtracting of their like terms. There is a great similarity between the operations with polynomials and denominate numbers. Compare the following examples:

1. Add 5 qt and 1 pt to 3 qt and 2 pt.

$$\begin{array}{r} 3\text{ qt} + 2\text{ pt} \\ \underline{5\text{ qt} + 1\text{ pt}} \\ 8\text{ qt} + 3\text{ pt} \end{array}$$

2. Add 5x + y to 3x + 2y.

$$\begin{array}{r} 3x + 2y \\ \underline{5x + y} \\ 8x + 3y \end{array}$$

One method of adding polynomials (shown in the above examples) is to place like terms in columns and to find the algebraic sum of the like terms. For example, to add 3a + b - 3c, 3b + c - d, and 2a + 4d, we would arrange the polynomials as follows:

$$\begin{array}{rrrr} 3a & + b & - 3c & \\ & 3b & + c & - d \\ 2a & & & + 4d \\ \hline 5a & + 4b & - 2c & + 3d \end{array}$$

Subtraction may be performed by using the same arrangement—that is, by placing terms of the subtrahend under the like terms of the minuend and carrying out the subtraction with due regard for sign. Remember, in subtraction the signs of all the terms of the subtrahend must first be mentally changed and then the process completed as in addition. For example, subtract 10a + b from 8a - 2b, as follows:

$$\begin{array}{rr} 8a & - 2b \\ 10a & + b \\ \hline -2a & - 3b \end{array}$$

Again, note the similarity between this type of subtraction and the subtraction of denominate numbers.

Addition and subtraction of polynomials also can be indicated with the aid of symbols of grouping. The rule regarding changes of sign when removing parentheses preceded by a minus sign automatically takes care of subtraction For example, to subtract 10a + b from 8a - 2b we can use the following arrangement:

$$\begin{aligned}(8a - 2b) - (10a + b) &= 8a - 2b - 10a - b \\ &= -2a - 3b\end{aligned}$$

Similarly, to add -3x + 2y to -4x - 5y, we ca write

$$\begin{aligned}(-3x + 2y) + (-4x - 5y) &= -3x + 2y - 4x - 5y \\ &= -7x - 3y\end{aligned}$$

Practice problems. Add as indicated, i each of the following problems:

1. $\begin{array}{r} 3a + b \\ \underline{2a + 5b} \end{array}$
2. $(6s^3t + 3s^2t + st + 5) + (s^3t - 5)$
3. 4a + b + c, a + c - d, and 3a + 2b + 2c
4. $\begin{array}{rrr} 4x & + 2y & \\ 3x & - y & + z \\ \underline{x} & & \underline{- z} \end{array}$

In problems 5 through 8, perform the indicated operations and combine like terms.

5. (2a + b) - (3a + 5b)
6. $(5x^3y + 3x^2y) - (x^3y)$
7. (x + 6) + (3x + 7)
8. $(4a^2 - b) - (2a^2 + b)$

Answers:

1. 5a + 6b
2. $7s^3t + 3s^2t + st$
3. 8a + 3b + 4c - d
4. 8x + y
5. -(a + 4b)
6. $4x^3y + 3x^2y$
7. 4x + 13
8. $2(a^2 - b)$

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

We can explain the multiplication of a polynomial by a monomial by using an arithmetic example. Let it be required to multiply the binomial expression, 7 - 2, by 4. We may write this 4 x (7 - 2) or simply 4(7 - 2). Now 7 - 2 = 5. Therefore, 4(7 - 2) = 4(5) = 20. Now, let us solve the problem a different way. Instead of subtracting first and then multiplying, let us multiply each term of the expression by 4 and then subtract. Thus, $4(7 - 2) = (4 \times 7) - (4 \times 2) = 20$. Both methods give the same result. The second method makes use of the distributive law of multiplication.

When there are literal parts in the expression to be multiplied, the first method cannot be used and the distributive method must be employed. This is illustrated in the following examples:

$$4(5 + a) = 20 + 4a$$

$$3(a + b) = 3a + 3b$$

$$ab(x + y - z) = abx + aby - abz$$

Thus, to multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial.

Practice problems. Multiply as indicated:

1. $2a(a - b)$
2. $4a^2(a^2 + 5a + 2)$
3. $-4x(-y - 3z)$
4. $2a^3(a^2 - ab)$

Answers:

1. $2a^2 - 2ab$
2. $4a^4 + 20a^3 + 8a^2$
3. $4xy + 12xz$
4. $2a^5 - 2a^4b$

MULTIPLICATION OF A POLYNOMIAL BY A POLYNOMIAL

As with the monomial multiplier, we explain the multiplication of a polynomial by a polynomial by use of an arithmetic example. To multiply $(3 + 2)(6 - 4)$, we could do the operation within the parentheses first and then multiply, as follows:

$$(3 + 2)(6 - 4) = (5)(2) = 10$$

However, thinking of the quantity $(3 + 2)$ as one term, we can use the method described for a monomial multiplier. That is, we can multiply each term of the multiplicand by the multiplier, $(3 + 2)$, with the following result:

$$(3 + 2)(6 - 4) = [(3 + 2) \times 6 - (3 + 2) \times 4]$$

Now considering each of the two resulting products separately, we note that each is a binomial multiplied by a monomial. The first is

$$(3 + 2)6 = (3 \times 6) + (2 \times 6)$$

and the second is

$$\begin{aligned} -(3 \times 2)4 &= - [(3 \times 4) + (2 \times 4)] \\ &= -(3 \times 4) - (2 \times 4) \end{aligned}$$

Thus we have the following result:

$$\begin{aligned} (3 + 2)(6 - 4) &= (3 \times 6) + (2 \times 6) \\ &\qquad - (3 \times 4) - (2 \times 4) \\ &= 18 + 12 - 12 - 8 \\ &= 10 \end{aligned}$$

The complete product is formed by multiplying each term of the multiplicand separately by each term of the multiplier and combining the results with due regard to signs.

Now let us apply this method in two examples involving literal numbers.

1. $(a + b)(m + n) = am + an + bm + bn$

2. $(2b + c)(r + s + 3t - u) = 2br + 2bs$ $+ 6bt - 2bu + cr + cs + 3ct - cu$

The rule governing these examples is stated as follows: The product of any two polynomials is found by multiplying each term of one by each term of the other and adding the results algebraically.

It is often convenient, especially when either of the expressions contains more than two terms, to place the polynomial with the fewer terms beneath the other polynomial and multiply term by term beginning at the left. Like terms of the partial products are placed one beneath the other to facilitate addition.

Suppose we wish to find the product of $3x^2 - 7x - 9$ and $2x - 3$. The procedure is

$$\begin{array}{rrrr} 3x^2 & - 7x & - 9 & \\ 2x & - 3 & & \\ \hline 6x^3 & - 14x^2 & - 18x & \\ & - 9x^2 & + 21x & + 27 \\ \hline 6x^3 & - 23x^2 & + 3x & + 27 \end{array}$$

Practice problems. In the following problems, multiply and combine like terms:

1. $(2a - 3)(a + 2)$
2. $(ax + b)(ax - b)$
3. $$\begin{array}{l} x^3 + 5x^2 - x + 2 \\ \underline{2x + 3 \qquad\qquad\quad} \end{array}$$
4. $$\begin{array}{l} 2a^2 + 5ab - b^2 \\ \underline{a + b \qquad\qquad\quad} \end{array}$$

Answers:

1. $2a^2 + a - 6$
2. $a^2x^2 - b^2$
3. $2x^4 + 13x^3 + 13x^2 + x + 6$
4. $2a^3 + 7a^2b + 4ab^2 - b^3$

SPECIAL PRODUCTS

The products of certain binomials occur frequently. It is convenient to remember the form of these products so that they can be written immediately without performing the complete multiplication process. We present four such special products as follows, and then show how each is derived:

1. Product of the sum and difference of two numbers.

EXAMPLE: $(x - y)(x + y) = x^2 - y^2$

2. Square the sum of two numbers.

EXAMPLE: $(x + y)^2 = x^2 + 2xy + y^2$

3. Square of the difference of two numbers.

EXAMPLE: $(x - y)^2 = x^2 - 2xy + y^2$

4. Product of two binomials having a common term.

EXAMPLE: $(x + a)(x + b) = x^2 + (a + b)x + ab$

Product of Sum and Difference

The product of the sum and difference of two numbers is equal to the square of the first number minus the square of the second number. If, for example, x - y is multiplied by x + y, the middle terms cancel one another. The result is the square of x minus the square of y, as shown in the following illustration:

$$\begin{array}{llll} x & - y & & \\ \underline{x} & \underline{+ y} & & \\ x^2 & - xy & & \\ \underline{} & \underline{+ xy} & \underline{- y^2} & \\ x^2 & & - y^2 & \end{array}$$

By keeping this rule in mind, the product of the sum and difference of two numbers can be written down immediately by writing the difference of the squares of the numbers. For example, consider the following three problems:

$$(x + 3)(x - 3) = x^2 - 3^2 = x^2 - 9$$

$$(5a + 2b)(5a - 2b) = (5a)^2 - (2b)^2 = 25a^2 - 4b^2$$

$$(7x + 4y)(7x - 4y) = 49x^2 - 16y^2$$

RATIONALIZING DENOMINATORS.— The product of the sum and difference of two numbers is useful in rationalizing a denominator that is a binomial. For example, in a fraction such as

$$\frac{2}{\sqrt{2} - 6}$$

the denominator can be altered so that no radical terms appear in it. (This process is called rationalizing.) The denominator must be multiplied by $\sqrt{2} + 6$, which is called the conjugate of $\sqrt{2} - 6$. Since the value of the original fraction would be changed if we multiplied only the denominator, our multiplier must be applied to both the numerator and the denominator. Multiplying the original fraction by

$$\frac{\sqrt{2} + 6}{\sqrt{2} + 6}$$

is, in effect, the same as multiplying it by 1.

The result of rationalizing the denominator of this fraction is as follows:

$$\begin{aligned} \frac{2}{\sqrt{2} - 6} \cdot \frac{\sqrt{2} + 6}{\sqrt{2} + 6} &= \frac{2(\sqrt{2} + 6)}{(\sqrt{2})^2 - 6^2} \\ &= \frac{2(\sqrt{2} + 6)}{2 - 36} \\ &= \frac{2(\sqrt{2} + 6)}{2(1 - 18)} \\ &= \frac{2(\sqrt{2} + 6)}{2(-17)} \\ &= \frac{\sqrt{2} + 6}{-17} \end{aligned}$$

MENTAL MULTIPLICATION.—The product of the sum and difference can be utilized to mentally multiply two numbers that differ from a multiple of 10 by the same amount, one greater and the other less. For example, 67 is 3 less than 70 while 73 is 3 more than 70. The product of 67 and 73 is then found as follows:

$$\begin{aligned} 67(73) &= (70 - 3)(70 + 3) \\ &= 70^2 - 3^2 = 4{,}900 - 9 = 4{,}891 \end{aligned}$$

Square of Sum or Difference

The square of the SUM of two numbers is equal to the square of the first number plus twice the product of the numbers plus the square of the second number. The square of the DIFFERENCE of the same two numbers has the same form, except that the sign of the middle term is negative.

These results are evident from multiplication. When x and y represent the two numbers, we obtain

$$\begin{array}{lll} x + y \\ \underline{x + y} \\ x^2 + xy \\ \underline{\quad + xy + y^2} \\ x^2 + 2xy + y^2 \end{array} \qquad \begin{array}{lll} x - y \\ \underline{x - y} \\ x^2 - xy \\ \underline{\quad - xy + y^2} \\ x^2 - 2xy + y^2 \end{array}$$

Applying this rule to the squares of the binomials $3a + 2b$ and $3a - 2b$, we have the following two cases:

1. $(3a + 2b)^2 = (3a)^2 + 2(3a)(2b) + (2b)^2$
   $= 9a^2 + 12ab + 4b^2$

2. $(3a - 2b)^2 = 9a^2 - 12ab + 4b^2$

The square of the sum or difference of two numbers is applicable to squaring a binomial that contains one or two irrational terms, as in the following examples:

1. $(\sqrt{3} + 8)^2 = (\sqrt{3})^2 + 2(8)(\sqrt{3}) + 64$
   $= 3 + 16\sqrt{3} + 64 = 67 + 16\sqrt{3}$

2. $(\sqrt{3} - 8)^2 = (\sqrt{3})^2 - 2(8)(\sqrt{3}) + 64$
   $= 3 - 16\sqrt{3} + 64 = 67 - 16\sqrt{3}$

3. $(\sqrt{5} + \sqrt{7})^2 = (\sqrt{5})^2 + 2\sqrt{5}\sqrt{7} + (\sqrt{7})^2$
   $= 5 + 2\sqrt{35} + 7 = 12 + 2\sqrt{35}$

4. $(\sqrt{5} - \sqrt{7})^2 = 12 - 2\sqrt{35}$

The square of the sum or difference of two numbers can be applied to the process of mentally squaring certain numbers. For example, 82^2 can be expressed as $(80 + 2)^2$ while 67^2 can be expressed as $(70 - 3)^2$. We find that

$$\begin{aligned} (80 + 2)^2 &= 80^2 + 2(80)(2) + 2^2 \\ &= 6{,}400 + 320 + 4 = 6{,}724 \end{aligned}$$

$$\begin{aligned} (70 - 3)^2 &= 70^2 - 2(70)(3) + 3^2 \\ &= 4{,}900 - 420 + 9 = 4{,}489 \end{aligned}$$

Binomials Having a Common Term

The binomials $x + 2$ and $x - 3$ have a common term, x. They have two unlike terms, +2 and -3. The product of these binomials is

$$\begin{array}{l} x \ + 2 \\ \underline{x \ - 3} \\ x^2 + 2x \\ \underline{\quad\ - 3x - 6} \\ x^2 - \ x - 6 \end{array}$$

Inspection of this product shows that it is obtained by squaring the common term, adding the sum of the unlike terms multiplied by the common term, and finally adding the product of the unlike terms.

Apply this rule to the product of $3y - 5$ and $3y + 4$. The common term is 3y; its square is $9y^2$. The sum of the unlike terms is $-5 + 4 = -1$; the sum of the unlike terms multiplied by the common term is $-3y$; and the product of the unlike terms is $-5(4) = -20$. The product of the two binomials is

$$(3y - 5)(3y + 4) = 9y^2 - 3y - 20$$

The product of two binomials having a common term is applicable to the multiplication of numbers like $\sqrt{3} + 7$ and $\sqrt{3} - 2$ which contain irrational terms. For example,

$$\begin{aligned} (\sqrt{3} + 7)(\sqrt{3} - 2) &= (\sqrt{3})^2 + 5\sqrt{3} - 14 \\ &= 3 + 5\sqrt{3} - 14 \\ &= -11 + 5\sqrt{3} \end{aligned}$$

Practice problems. In problems 1 through 4, multiply and combine terms. In 5 through 8, simplify by using special products.

1. $(x + 4)(x + 2)$
2. $(\sqrt{a} - b)^2$
3. $(7a + 4b)(7a - 4b)$
4. $(ax + y)^2$
5. $\dfrac{2}{\sqrt{2} - 2}$
6. $48(52)$
7. $(\sqrt{3} + 7)^2$
8. $(73)^2$

Answers:

1. $x^2 + 6x + 8$
2. $a - 2b\sqrt{a} + b^2$
3. $49a^2 - 16b^2$
4. $a^2x^2 + 2axy + y^2$
5. $-(\sqrt{2} + 2)$
6. $(50 - 2)(50 + 2)$ $= 2496$
7. $52 + 14\sqrt{3}$
8. $(70 + 3)(70 + 3)$ $= 5329$

DIVISION OF A POLYNOMIAL BY A MONOMIAL

Division, like multiplication, may be distributive. Consider, for example, the problem $(4 + 6 - 2) \div 2$, which may be solved by adding the numbers within the parentheses and then dividing the total by 2. Thus,

$$\frac{4 + 6 - 2}{2} = \frac{8}{2} = 4$$

Now notice that the problem may also be solved distributively.

$$\begin{aligned}\frac{4 + 6 - 2}{2} &= \frac{4}{2} + \frac{6}{2} - \frac{2}{2} \\ &= 2 + 3 - 1 \\ &= 4\end{aligned}$$

CAUTION: Do not confuse problems of the type just described with another type which is similar in appearance but not in final result. For example, in a problem such as $2 \div (4 + 6 - 2)$ the beginner is tempted to divide 2 successively by 4, then 6, and then -2, as follows:

$$\frac{2}{4 + 6 - 2} \neq \frac{2}{4} + \frac{2}{6} - \frac{2}{2}$$

Notice that we have canceled the "equals" sign, because $2 \div 8$ is obviously not equal to $1/2 + 2/6 - 1$. The distributive method applies only in those cases in which several different numerators are to be used with the same denominator

When literal numbers are present in an expression, the distributive method must be used, as in the following two problems:

$$1.\quad \begin{aligned}\frac{2ax + aby + a}{a} &= \frac{2ax}{a} + \frac{aby}{a} + \frac{a}{a} \\ &= 2x + by + 1\end{aligned}$$

$$2.\quad \begin{aligned}\frac{18ab^2 - 12bc}{6b} &= \frac{18ab^2}{6b} - \frac{12bc}{6b} \\ &= 3ab - 2c\end{aligned}$$

Quite often this division may be done mentally, and the intermediate steps need not be written out.

DIVISION OF A POLYNOMIAL BY A POLYNOMIAL

Division of one polynomial by another proceeds as follows:

1. Arrange both the dividend and the divisor in either descending or ascending powers of the same letter.
2. Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.
3. Multiply the complete divisor by the quotient just obtained, write the terms of the product under the like terms of the dividend, and subtract this expression from the dividend.
4. Consider the remainder as a new dividend and repeat steps 1, 2, and 3.

EXAMPLE:

$$(10x^3 - 7x^2y - 16xy^2 + 12y^3) \div (5x - 6y)$$

SOLUTION:

$$\begin{array}{rl}
 & 2x^2 + xy - 2y^2 \\
5x - 6y & \overline{)\,10x^3 - 7x^2y - 16xy^2 + 12y^3} \\
 & \underline{10x^3 - 12x^2y} \\
 & \qquad\quad 5x^2y - 16xy^2 \\
 & \qquad\quad \underline{5x^2y - 6xy^2} \\
 & \qquad\qquad\quad -10xy^2 + 12y^3 \\
 & \qquad\qquad\quad \underline{\underline{-10xy^2 + 12y^3}}
\end{array}$$

In the example just shown, we began by dividing the first term, $10x^3$, of the dividend by the first term, $5x$, of the divisor. The result is $2x^2$. This is the first term of the quotient.

Next, we multiply the divisor by $2x^2$ and subtract this product from the dividend. Use the remainder as a new dividend. Get the second term, xy, in the quotient by dividing the first term, $5x^2y$, of the new dividend by the first term, 5x, of the divisor. Multiply the divisor by xy and again subtract from the dividend. Continue the process until the remainder is zero or is of a degree lower than the divisor. In the example being considered, the remainder is zero (indicated by the double line at the bottom). The quotient is $2x^2 + xy - 2y^2$.

The following long division problem is an example in which a remainder is produced:

$$\begin{array}{rl}
 & x^2 - x + 3 \\
x + 3 & \overline{\big) x^3 + 2x^2 \qquad + 5} \\
 & \underline{x^3 + 3x^2} \\
 & \qquad - x^2 \\
 & \qquad \underline{- x^2 - 3x} \\
 & \qquad\qquad 3x + 5 \\
 & \qquad\qquad \underline{3x + 9} \\
 & \qquad\qquad\quad - 4
\end{array}$$

The remainder is -4.

Notice that the term -3x in the second step of this problem is subtracted from zero, since there is no term containing x in the dividend. When writing down a dividend for long division, leave spaces for missing terms which may enter during the long division process.

In arithmetic, division problems are often arranged as follows, in order to emphasize the relationship between the remainder and the divisor:

$$\frac{5}{2} = 2 + \frac{1}{2}$$

This same type of arrangement is used in algebra. For example, in the problem just shown, the results could be written as follows:

$$\frac{x^3 + 2x^2 + 5}{x + 3} = x^2 - x + 3 - \frac{4}{x + 3}$$

Remember, before dividing polynomials arrange the terms in the dividend and divisor according to either descending or ascending powers of one of the literal numbers. When only one literal number occurs, the terms are usually arranged in order of descending powers.

For example, in the polynomial $2x^2 + 4x^3 + 5 - 7x$ the highest power among the literal terms is x^3. If the terms are arranged according to descending powers of x, the term in x^3 should appear first. The x^3 term should be followed by the x^2 term, the x term, and finally the constant term. The polynomial arranged according to descending powers of x is $4x^3 + 2x^2 - 7x + 5$.

Suppose that $4ab + b^2 + 15a^2$ is to be divided by $3a + 2b$. Since 3a can be divided evenly into $15a^2$, arrange the terms according to descending powers of a. The dividend takes the form

$$15a^2 + 4ab + b^2$$

Synthetic Division

Synthetic division is a shorthand method of dividing a polynomial by a binomial of the form x - a. For example, if $3x^4 + 2x^3 + 2x^2 - x - 6$ is to be divided by x - 1, the long form would be as follows:

$$\begin{array}{rl}
 & \qquad 3x^3 + 5x^2 + 7x + 6 \\
x - 1 & \overline{\big) 3x^4 + 2x^3 + 2x^2 - x - 6} \\
 & \underline{3x^4 - 3x^3} \\
 & \qquad + 5x^3 + 2x^2 \\
 & \qquad \underline{+ 5x^3 - 5x^2} \\
 & \qquad\qquad + 7x^2 - x \\
 & \qquad\qquad \underline{+ 7x^2 - 7x} \\
 & \qquad\qquad\qquad + 6x - 6 \\
 & \qquad\qquad\qquad \underline{\underline{+ 6x - 6}}
\end{array}$$

Notice that every alternate line of work in this example contains a term which duplicates the one above it. Furthermore, when the subtraction is completed in each step, these duplicated terms cancel each other and thus have no effect on the final result. Another unnecessary duplication results when terms from the dividend are brought down and rewritten prior to subtraction. By omitting these duplications, the work may be condensed as follows:

$$\begin{array}{r|rrrrr}
 & 3x^3 & +5x^2 & +7x & +6 & \\
\hline
x - 1 & 3x^4 & +2x^3 & +2x^2 & -x & -6 \\
 & & -3x^3 & -5x^2 & -7x & -6 \\
\hline
 & & +5x^3 & +7x^2 & +6x & 0
\end{array}$$

The coefficients of the dividend and the constant term of the divisor determine the results of each successive step of multiplication and subtraction. Therefore, we may condense still further by writing only the nonliteral factors, as follows:

$$\begin{array}{r|rrrrr} & 3 & +5 & +7 & +6 & \\ \hline -1 & 3 & +2 & +2 & -1 & -6 \\ & & -3 & -5 & -7 & -6 \\ \hline & 3 & +5 & +7 & +6 & 0 \end{array}$$

Notice that if the coefficient of the first term in the dividend is brought down to the last line, then the numbers in the last line are the same as the coefficients of the terms in the quotient. Thus we do not really need to write a separate line of coefficients to represent the quotient. Instead, we bring down the first coefficient of the dividend and make the subtraction "subtotals" serve as coefficients for the rest of the quotient, as follows:

$$\begin{array}{r|rrrrr} x - 1 & 3 & 2 & 2 & -1 & -6 \\ & & -3 & -5 & -7 & -6 \\ \hline & 3 & 5 & 7 & 6 & 0 \end{array}$$

The unnecessary writing of plus signs is also eliminated here.

The use of synthetic division is limited to divisors of the form x - a, in which the degree of x is 1. Thus the degree of each term in the quotient is 1 less than the degree of the corresponding term in the dividend. The quotient in this example is as follows:

$$3x^3 + 5x^2 + 7x + 6$$

The sequence of operations in synthetic division may be summarized as follows, using as an example the division of $3x - 4x^2 + x^4 - 3$ by $x + 2$:

$$\begin{array}{r|rrrrr} 2 & 1 & 0 & -4 & 3 & -3 \\ & & 2 & -4 & 0 & 6 \\ \hline & 1 & -2 & 0 & 3 & -9 \end{array}$$

First, rearrange the terms of the dividend in descending powers of x. The dividend then becomes $x^4 - 4x^2 + 3x - 3$, with 1 understood as the coefficient of the first term. No x^3 term appears in the polynomial, but we supply a zero as a place holder for the x^3 position.

Second, bring down the 1 and multiply it by the +2 of the divisor. Place the result under the zero, and subtract. Multiply the result (-2) by the +2 of the divisor, place the product under the -4 of the dividend, and subtract. Continue this process, finally obtaining $x^3 - 2x^2 + 3$ as the quotient. The remainder is -9.

Practice problems. In the following problems, perform the indicated operations. In 4, 5, and 6, first use synthetic division and then check your work by long division:

1. $(a^3 - 3a^2 + a) \div a$
2. $\dfrac{x^6 - 7x^5 + 4x^4}{x^2}$
3. $(10x^3 - 7x^2y - 16xy^2 + 12y^3)$
   $\div (2x^2 + xy - 2y^2)$
4. $(x^2 + 11x + 30) \div (x + 6)$
5. $(12 + x^2 - 7x) \div (x - 3)$
6. $(a^2 - 11a + 30) \div (a - 5)$

Answers:

1. $a^2 - 3a + 1$
2. $x^4 - 7x^3 + 4x^2$
3. $5x - 6y$
4. $x + 5$
5. $x - 4$
6. $a - 6$

CHAPTER 10

FACTORING POLYNOMIALS

A factor of a quantity N, as defined in chapter 2 of this course, is any expression which can be divided into N without producing a remainder. Thus 2 and 3 are factors of 6, and the factors of 5x are 5 and x. Conversely, when all of the factors of N are multiplied together, the product is N. This definition is extended to include polynomials.

The factors of a polynomial are two or more expressions which, when multiplied together, give the polynomial as a product. For example, 3, x, and $x^2 - 4$ are factors of $3x^3 - 12x$, as the following equation shows:

$$(3)(x)(x^2 - 4) = 3x^3 - 12x$$

The factors 3 and x, which are common to both terms of the polynomial $3x^3 - 12x$, are called COMMON FACTORS.

The distributive principle, mentioned in chapters 3 and 9 of this course, is an important part of the concept of factoring. It may be stated as follows:

If the sum of two or more quantities is multiplied by a third quantity, the product is found by applying the multiplier to each of the original quantities separately and summing the resulting expressions. It is this principle which allows us to separate common factors from the terms of a polynomial.

Just as with numbers, an algebraic expression is a prime factor if it has no other factors except itself and 1. The factor $x^2 - 4$ is not prime, since it can be separated into $x - 2$ and $x + 2$. The factors $x - 2$ and $x + 2$ are both prime factors, since they cannot be separated into other factors.

The process of finding the factors of a polynomial is called FACTORING. An expression is said to be factored completely when it has been separated into its prime factors. The polynomial $3x^3 - 12x$ is factored completely as follows:

$$3x^3 - 12x = 3x(x - 2)(x + 2)$$

COMMON FACTORS

Factoring any polynomial begins with the removal of common factors. Notice that "removal" of a factor does not mean discarding it. To remove a factor is to insert parentheses and move the factor outside the parentheses as a common multiplier. The removal of common factors proceeds as follows:

1. Inspect the polynomial and find the factors which are common to all terms. These common factors, multiplied together, comprise the "largest common factor."
2. Mentally divide each term of the polynomial by the largest common factor and write the quotients within a set of parentheses.
3. Write the largest common factor outside the parentheses as a common multiplier.

For example, the expression $x^2y - xy^2$ contains xy as a factor of each term. Therefore, it is factored as follows:

$$x^2y - xy^2 = xy(x - y)$$

Other examples of factoring by the removal of common factors are found in the following expressions:

$$6m^4n + 3m^3n^2 - 3m^2n^3 = 3m^2n(2m^2 + mn - n^2)$$

$$-5z^2 - 15z = -5z(z + 3)$$

$$7x - 7y + 7z = 7(x - y + z)$$

In selecting common factors, always remove as many factors as possible from each term in order to factor completely. For example, x is a factor of $3ax^2 - 3ax$, so that $3ax^2 - 3ax$ is equal to $x(3ax - 3a)$. However, 3 and a are also factors. Thus the largest common factor is 3ax. When factored completely, the expression is as follows:

$$3ax^2 - 3ax = 3ax(x - 1)$$

Practice problems: Remove the common factors:

1. $y^2 - y$
2. $a^3b^2 - a^2b^2$
3. $2b^3 - 8b^2 - 6b$
4. $6mn^2 + 30m^2n$
5. $\frac{2}{3}x - \frac{1}{3}y + \frac{1}{3}$

Answers:

1. $y(y - 1)$
2. $a^2b^2(a - 1)$
3. $2b(b^2 - 4b - 3)$
4. $6mn(n + 5m)$
5. $\frac{1}{3}(2x - y + 1)$

LITERAL EXPONENTS

It is frequently necessary to remove common factors involving literal exponents; that is, exponents composed of letters rather than numbers. A typical expression involving literal exponents is $x^{2a} + x^a$, in which x^a is a common factor. The factored form is $x^a(x^a + 1)$. Another example of this type is $a^{m+n} + 2a^m$. Remember that a^{m+n} is equivalent to $a^m \cdot a^n$. Thus the factored form is as follows:

$$\begin{aligned} a^{m+n} + 2a^m &= a^m \cdot a^n + 2a^m \\ &= a^m(a^n + 2) \end{aligned}$$

BINOMIAL FORM

The distinctions between monomial, binomial, and trinomial factors are discussed in detail in chapter 9 of this course. An expression such as $a(x + y) + b(x + y)$ has a common factor in binomial form. The factor $(x + y)$ can be removed from both terms, with the following result:

$$a(x + y) + b(x + y) = (x + y)(a + b)$$

Sometimes it is easier to see this if a single letter is substituted temporarily for the binomial. Thus, let $(x + y) = n$, so that $a(x + y) + b(x + y)$ reduces to $(an + bn)$. The factored form is $n(a + b)$, which becomes $(x + y)(a + b)$ when n is replaced by its equal, $(x + y)$.

Another form of this type is $x(y - z) - w(z - y)$. Notice that this expression could be factored easily if the binomial in the second term were $(y - z)$. We can show that $-w(z - y)$ is equivalent to $+w(y - z)$, as follows:

$$\begin{aligned} -w(z - y) &= -w\ [(-1) \cdot (-1) \cdot z + (-1) \cdot y] \\ &= -w\ \{(-1)\ [(-1)\ z + y]\} \\ &= (-w)(-1)\ [-z + y] \\ &= +w(y - z) \end{aligned}$$

Substituting $+w(y - z)$ for $-w(z - y)$ in the original expression, we may now factor as follows:

$$\begin{aligned} x(y - z) - w(z - y) &= x(y - z) + w(y - z) \\ &= (y - z)(x + w) \end{aligned}$$

In factoring an expression such as $ax + bx + ay + by$, common monomial factors are removed first, as follows:

$$ax + bx + ay + by = x(a + b) + y(a + b)$$

Having removed the common monomial factors, we then remove the common binomial factor to obtain $(a + b)(x + y)$.

Notice that we could have rewritten the expression as $ax + ay + bx + by$, based on the commutative law of addition, which states that the sum of two or more terms is the same regardless of the order in which they are arranged. The first step in factoring would then produce $a(x + y) + b(x + y)$ and the final form would be $(x + y)(a + b)$. This is equivalent to $(a + b)(x + y)$, by the commutative law of multiplication, which states that the product of two or more factors is the same regardless of the order in which they are arranged.

Practice problems. Factor each of the following:

1. $x^{3a} + 3x^{2a}$
2. $xy^2 + y + x^2y + x$
3. $e^x + 4e^{4x}$
4. $7(x^2 + y^2) - 3z(x^2 + y^2)$
5. $a^2 + ab - ac - cb$
6. $\frac{1}{2}e^2r - \frac{1}{6}er^2$
7. $a^{x+2} + a^2$
8. $xy - 3x - 2y + 6$

Answers:

1. $x^{2a}(x^a + 3)$
2. $(xy + 1)(x + y)$
3. $e^x(1 + 4e^{3x})$
4. $(x^2 + y^2)(7 - 3z)$
5. $(a + b)(a - c)$
6. $\frac{1}{2}er(e - \frac{1}{3}r)$
7. $a^2(a^x + 1)$
8. $(y - 3)(x - 2)$

BINOMIAL FACTORS

After any common factor has been removed from a polynomial, the remaining polynomial factor must be examined further for other factors. Skill in factoring is principally the ability to recognize certain types of products such as the square of a sum or difference. Therefore, it is important to be familiar with the special products discussed in chapter 9.

DIFFERENCE OF TWO SQUARES

In chapter 9 we learned that the product of the sum and difference of two numbers is the difference of their squares. Thus, $(a + b)(a - b) = a^2 - b^2$. Conversely, if a binomial is the difference of two squares, its factors are the sum and difference of the square roots. For example, in $9a^2 - 4b^2$ both $9a^2$ and $4b^2$ are perfect squares. The square roots are $3a$ and $2b$, respectively. Connect these square roots with a plus sign to get one factor of $9a^2 - 4b^2$ and with a minus sign to get the other factor. The two binomial factors are $3a - 2b$ and $3a + 2b$. Therefore, factored completely, the binomial can be written as follows:

$$9a^2 - 4b^2 = (3a - 2b)(3a + 2b)$$

We may check to see if these factors are correct by multiplying them together to see if their product is the original binomial.

The expression $20x^3y - 5xy^3$ reduces to the difference of two squares after the common factor $5xy$ is removed. Completely factored, this expression produces the following:

$$\begin{aligned} 20x^3y - 5xy^3 &= 5xy(4x^2 - y^2) \\ &= 5xy(2x - y)(2x + y) \end{aligned}$$

Other examples that show the difference of two squares in factored form are as follows:

$$\begin{aligned} 49 - 16 &= (7 + 4)(7 - 4) \\ 16a^2 - 4x^2 &= 4(4a^2 - x^2) \\ &= 4(2a + x)(2a - x) \\ 4x^2y - 9y &= y(4x^2 - 9) \\ &= y(2x + 3)(2x - 3) \end{aligned}$$

Practice problems: Factor each of the following:

1. $a^2 - b^2$
2. $b^2 - 9$
3. $a^2b^2 - 1$
4. $a^2 - 144$
5. $x^2 - y^2$
6. $y^2 - 36$
7. $1 - 4y^2$
8. $9a^2 - 16$

Answers:

1. $(a + b)(a - b)$
2. $(b + 3)(b - 3)$
3. $(ab + 1)(ab - 1)$
4. $(a + 12)(a - 12)$
5. $(x + y)(x - y)$
6. $(y + 6)(y - 6)$
7. $(1 + 2y)(1 - 2y)$
8. $(3a + 4)(3a - 4)$

SPECIAL BINOMIAL FORMS

Special cases involving binomial expressions are frequently encountered. All such expressions may be factored by reference to general formulas, but these formulas are beyond the scope of this course. For our purposes, analysis of some special cases will be sufficient.

Even Exponents

When the exponents on both terms of the binomial are even, the expression may be treated as the sum or difference of two squares. For example, $x^6 - y^6$ can be rewritten as $(x^3)^2 - (y^3)^2$ which results in the following factored form:

$$x^6 - y^6 = (x^3 - y^3)(x^3 + y^3)$$

In general, a binomial with even exponents has the form $x^{2m} \pm y^{2n}$, since all even numbers have 2 as a factor. If the connecting sign is positive, the expression may not be factorable; for example, $x^2 + y^2$, $x^4 + y^4$, and $x^8 + y^8$ are all nonfactorable binomials. If the connecting sign is negative, a binomial with even exponents is factorable as follows:

$$x^{2m} - y^{2n} = (x^m - y^n)(x^m + y^n)$$

A special case which is particularly important because it occurs so often is the binomial which has the numeral 1 as one of its terms. For example, $x^4 - 1$ is factorable as the difference of two squares, as follows:

$$\begin{aligned} x^4 - 1 &= (x^2 - 1)(x^2 + 1) \\ &= (x - 1)(x + 1)(x^2 + 1) \end{aligned}$$

Odd Exponents

Two special cases involving odd exponents are of particular importance. These are the sum of two cubes and the difference of two cubes. Examples of the sum and difference of two cubes, showing their factored forms, are as follows:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Notice that each of these factored forms involves a first degree binomial factor ((x + y) in the first case and (x - y) in the second). The connecting sign in the first degree binomial factor corresponds to the connecting sign in the original unfactored binomial.

We are now in a position to give the completely factored form of $x^6 - y^6$, as follows:

$$\begin{aligned} x^6 - y^6 &= (x^3 - y^3)(x^3 + y^3) \\ &= (x - y)(x^2 + xy + y^2) \\ &\qquad (x + y)(x^2 - xy + y^2) \end{aligned}$$

In general, (x + y) is a factor of $(x^n + y^n)$ if n is odd. If n is even, $(x^n + y^n)$ is not factorable unless it can be expressed as the sum of two cubes. When the connecting sign is negative, the binomial is always factorable if n is a whole number greater than 1. That is, (x - y) is a factor of $(x^n - y^n)$ for both odd and even values of n.

The special case in which one of the terms of the binomial is the numeral 1 occurs frequently. An example of this is $x^3 + 1$, which is factorable as the sum of two cubes, as follows:

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

In a similar manner, $1 + x^6$ can be treated as the sum of two cubes and factored as follows:

$$\begin{aligned} 1 + x^6 &= 1 + (x^2)^3 \\ &= (1 + x^2)(1 - x^2 + x^4) \end{aligned}$$

Practice problems. In each of the following problems, factor completely:

1. $x^4 - y^4$
2. $m^3 + n^3$
3. $x^6 - y^6$
4. $x^3 - y^3$
5. $a^9 - b^9$
6. $x^{2a} - y^{2b}$
7. $1 - x^4$
8. $x^6 + 1$
9. $1 - x^3$

Answers:

1. $(x + y)(x - y)(x^2 + y^2)$
2. $(m + n)(m^2 - mn + n^2)$
3. $(x + y)(x - y)(x^2 + xy + y^2)(x^2 - xy + y^2)$
4. $(x - y)(x^2 + xy + y^2)$
5. $(a - b)(a^2 + ab + b^2)(a^6 + a^3b^3 + b^6)$
6. $(x^a - y^b)(x^a + y^b)$
7. $(1 + x^2)(1 - x)(1 + x)$
8. $(x^2 + 1)(x^4 - x^2 + 1)$
9. $(1 - x)(1 + x + x^2)$

TRINOMIAL SQUARES

A trinomial that is the square of a binomial is called a TRINOMIAL SQUARE. Trinomials that are perfect squares factor into either the square of a sum or the square of a difference. Recalling that $(x + y)^2 = x^2 + 2xy + y^2$ and $(x - y)^2 = x^2 - 2xy + y^2$, the form of a trinomial square is apparent. The first term and the last term are perfect squares and their signs are positive. The middle term is twice the product of the square roots of these two numbers. The sign of the middle term is plus if a sum has been squared; it is minus if a difference has been squared.

The polynomial $16x^2 - 8xy + y^2$ is a trinomial in which the first term, $16x^2$, and the last term, y^2, are perfect squares with positive signs. The square roots are 4x and y. Twice the product of these square roots is 2(4x)(y) = 8xy. The middle term is preceded by a minus sign indicating that a difference has been squared. In factored form this trinomial is as follows:

$$16x^2 - 8xy + y^2 = (4x - y)^2$$

To factor the trinomial, we simply take the square roots of the end terms and join them with a plus sign if the middle term is preceded by a plus or with a minus if the middle term is preceded by a minus.

The terms of a trinomial may appear in any order. Thus, $8xy + y^2 + 16x^2$ is a trinomial square and may be factored as follows:

$$8xy + y^2 + 16x^2 = 16x^2 + 8xy + y^2 = (4x + y)^2$$

Practice problems. Among the following expressions, factor those which are trinomial squares:

1. $y^2 - 8y + 16$
2. $16y^2 + 30x + 9$
3. $36 + 12x + x^2$
4. $a^2 + 2ab + b^2$
5. $12y + 9y^2 - 4$
6. $4x^2 + y^2 + 4xy$
7. $9 - 6cd + c^2d^2$
8. $x^4 + 4x^2 + 4$

Answers:

1. $(y - 4)^2$
2. Not a trinomial square
3. $(6 + x)^2$
4. $(a + b)^2$
5. Not a trinomial square
6. $(2x + y)^2$
7. $(3 - cd)^2$
8. $(x^2 + 2)^2$

SUPPLYING THE MISSING TERM

Skill in recognizing trinomial squares may be improved by practicing the solution of problems which require supplying a missing term. For example, the expression $y^2 + (?) + 16$ can be made to form a perfect trinomial square by supplying the correct term to fill the parentheses.

The middle term must be twice the product of the square roots of the two perfect square terms; that is, $(2)(4)(y)$, or $8y$. Check: $y^2 + 8y + 16 = (y + 4)^2$. The missing term is $8y$.

Suppose that we wish to supply the missing term in $16x^2 + 24xy + (?)$ so that the three terms will form a perfect trinomial square. The square root of the first term is $4x$. One-half the middle term is $12xy$. Divide $12xy$ by $4x$. The result is $3y$ which is the square root of the last term. Thus, our missing term is $9y^2$. Checking, we find that $(4x + 3y)^2 = 16x^2 + 24xy + 9y^2$.

Practice problems. In each of the following problems, supply the missing term to form a perfect trinomial square:

1. $x^2 + (?) + y^2$
2. $t^2 + (?) + 25$
3. $9a^2 - (?) + 25b^2$
4. $4m^2 + 16m + (?)$
5. $x^2 + 4x + (?)$
6. $c^2 - 6cd + (?)$

Answers:

1. $2xy$
2. $10t$
3. $30ab$
4. 16
5. 4
6. $9d^2$

OTHER TRINOMIALS

It is sometimes possible to factor trinomials that are not perfect squares. Following are some examples of such trinomials, and the expressions of which they are products:

1. $(x + 3)(x + 4) = x^2 + 7x + 12$
2. $(x - 3)(x - 4) = x^2 - 7x + 12$
3. $(x - 3)(x + 4) = x^2 + x - 12$
4. $(x + 3)(x - 4) = x^2 - x - 12$

It is apparent that trinomials like these may be factored into binomials as shown. Notice how the trinomial in each of the preceding examples is formed. The first term is the square of the common term of the binomial factors. The second term is the algebraic sum of their unlike terms times their common term. The third term is the product of their unlike terms.

Such trinomials may be factored as the product of two binomials if there are two numbers such that their algebraic sum is the coefficient of the middle term and their product is the last term.

For example, let us factor the expression $x^2 - 12x + 32$. If the expression is factorable, there will be a common term, x, in each of the binomial factors. We begin factoring by placing this term within each set of parentheses, as follows:

$$(x \qquad)(x \qquad)$$

Next, we must find the other terms that are to go in the parentheses. They will be two numbers such that their algebraic sum is -12 and their product is +32. We see that -8 and -4 satisfy the conditions. Thus, the following expression results:

$$x^2 - 12x + 32 = (x - 8)(x - 4)$$

It is of value in factoring to note some useful facts about trinomials. If both the second and third terms of the trinomial are positive, the signs of the terms to be found are positive as in example 1 of this section. If the second term is negative and the last is positive, both terms to be found will be negative as in example 2. If the third term of the trinomial is negative, one of the terms to be found is positive and the other is negative as in examples 3 and 4. Concerning this last case, if the second term is positive as in example 3, the positive term in the factors has the greater numerical value. If the second term is negative as in example 4, the negative term in the factors has the greater numerical value.

It should be remembered that not all trinomials are factorable. For example, $x^2 + 4x + 2$ cannot be factored since there are no two rational numbers whose product is 2 and whose sum is 4.

Practice problems. Factor completely, in the following problems:

1. $y^2 + 15y + 50$
2. $y^2 - 2y - 24$
3. $x^2 + 8x - 48$
4. $x^2 - 4x - 60$
5. $x^2 - 12x - 45$
6. $x^2 - 15x + 56$
7. $x^2 + 2x - 48$
8. $x^2 + 14x + 24$

Answers:

1. $(y + 5)(y + 10)$
2. $(y - 6)(y + 4)$
3. $(x + 12)(x - 4)$
4. $(x - 10)(x + 6)$
5. $(x - 15)(x + 3)$
6. $(x - 7)(x - 8)$
7. $(x - 6)(x + 8)$
8. $(x + 12)(x + 2)$

Thus far we have considered only those expressions in which the coefficient of the first term is 1. When the coefficient of the first term is other than 1, the expression can be factored as shown in the following example:

$$6x^2 - x - 2 = (2x + 1)(3x - 2)$$

Although this result can be obtained by the trial and error method, the following procedure saves time and effort. First, find two numbers whose sum is the coefficient of the second term (-1 in this example) and whose product is equal to the product of the third term and the coefficient of the first term (in this example, (6)(-2) or -12). By inspection, the desired numbers are found to be -4 and +3. Using these two numbers as coefficients for x, we can rewrite the original expression as $6x^2 - 4x + 3x - 2$ and factor as follows:

$$\begin{aligned} 6x^2 - 4x + 3x - 2 &= 2x(3x - 2) + 1(3x - 2) \\ &= (2x + 1)(3x - 2) \end{aligned}$$

Practice problems. Factor completely, in the following problems:

1. $2x^2 + 13x + 21$
2. $16x^2 + 26x + 3$
3. $15x^2 - 16x - 7$
4. $12x^2 - 8x - 15$

Answers:

1. $(2x + 7)(x + 3)$
2. $(2x + 3)(8x + 1)$
3. $(3x + 1)(5x - 7)$
4. $(6x + 5)(2x - 3)$

REDUCING FRACTIONS TO LOWEST TERMS

There are many useful applications of factoring. One of the most important is that of simplifying algebraic fractions. Fractions that contain algebraic expressions in the numerator or denominator, or both, can be reduced to lower terms, if there are factors common to numerator and denominator. If the terms of a fraction are monomials, common factors are immediately apparent, as in the following expression:

$$\frac{3x^2y}{6xy} = \frac{3xy(x)}{3xy(2)} = \frac{x}{2}$$

If the terms of a fraction are polynomials, the polynomials must be factored in order to recognize the existence of common factors, as in the following two examples:

$$1.\ \frac{a - b}{a^2 - 2ab + b^2} = \frac{a - b}{(a - b)(a - b)} = \frac{1}{(a - b)}$$

$$2.\ \frac{4x^2 - 9}{6x^2 - 9x} = \frac{(2x + 3)(2x - 3)}{3x(2x - 3)} = \frac{(2x + 3)}{3x}$$

Notice that without the valuable process of factoring, we would be forced to use the fractions in their more complicated form. When there are factors common to both numerator and denominator, it is obviously more practical to cancel them (first using the factoring process) before proceeding.

Practice problems. Reduce to lowest terms in each of the following:

1. $\dfrac{12}{6x + 12}$
2. $\dfrac{a^2 - b^2}{a^2 - 2ab + b^2}$
3. $\dfrac{y^2 - 14y + 45}{y^2 - 8y - 9}$
4. $\dfrac{y^2 - 25}{y^2 - 8y + 15}$
5. $\dfrac{a^2 - 5a - 24}{a^2 - 64}$
6. $\dfrac{4x^2y - 9y}{4x^2 + 12x + 9}$

Answers:

1. $\frac{2}{x + 2}$
2. $\frac{a + b}{a - b}$
3. $\frac{y - 5}{y + 1}$
4. $\frac{y + 5}{y - 3}$
5. $\frac{a + 3}{a + 8}$
6. $\frac{y(2x - 3)}{2x + 3}$

OPERATIONS INVOLVING FRACTIONS

Addition, subtraction, multiplication, and division operations involving algebraic fractions are often simplified by means of factoring, whereas they would be quite complicated without the use of factoring.

MULTIPLYING FRACTIONS

Multiplication of fractions that contain polynomials is similar to multiplication of fractions that contain only arithmetic numbers. If this fact is kept in mind, the student will have little difficulty in mastering multiplication in algebra. For instance, we recall that to multiply a fraction by a whole number, we simply multiply the numerator by the whole number. This is illustrated in the following example:

Arithmetic: $4 \times \frac{3}{17} = \frac{12}{17}$

Algebra: $(x - 4) \cdot \frac{3}{x^2 - 5} = \frac{3x - 12}{x^2 - 5}$

Sometimes the work may be simplified by factoring and canceling before carrying out the multiplication. The following example illustrates this:

$$(2a - 8) \cdot \frac{3}{a^2 - 8a + 16} = \frac{2\cancel{(a - 4)}}{1} \cdot \frac{3}{(a - 4)\cancel{(a - 4)}}$$

$$= \frac{2(3)}{a - 4} = \frac{6}{a - 4}$$

When the multiplier is a fraction, the rules of arithmetic remain applicable—that is, multiply numerators together and denominators together. This is illustrated as follows:

Arithmetic: $\frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$

Algebra: $\frac{a + b}{a - b} \cdot \frac{a}{a - b} = \frac{a(a + b)}{(a - b)^2}$

Where possible, the work may be considerably reduced by factoring, canceling, and then carrying out the multiplication, as in the following example:

$$\frac{x^2 - 2x + 1}{x^2 - 9} \cdot \frac{x^2 + x - 6}{x^2 - 1}$$

$$= \frac{(x - 1)\cancel{(x - 1)}}{\cancel{(x + 3)}(x - 3)} \cdot \frac{\cancel{(x + 3)}(x - 2)}{(x + 1)\cancel{(x - 1)}}$$

$$= \frac{(x - 1)(x - 2)}{(x - 3)(x + 1)} = \frac{x^2 - 3x + 2}{x^2 - 2x - 3}$$

Although the factors may be multiplied to form two trinomials as shown, it is usually sufficient to leave the answer in factored form.

Practice problems. In the following problems, multiply as indicated:

1. $5a^2 \cdot \frac{3b}{a + b}$
2. $\frac{x + y}{x^2} \cdot \frac{x - y}{x - 1}$
3. $\frac{a^2 + 2ab + b^2}{a^2 - b^2} \cdot \frac{6a}{3a + 3b}$
4. $\frac{a - 1}{2a^2 + 4a + 2} \cdot \frac{(a + 1)^2}{a - 1}$

Answers:

1. $\frac{15a^2b}{a + b}$
2. $\frac{x^2 - y^2}{x^3 - x^2}$
3. $\frac{6a}{3(a - b)}$
4. $\frac{1}{2}$

DIVIDING FRACTIONS

The rules of arithmetic apply to the division of algebraic fractions; as in arithmetic, simply invert the divisor and multiply, as follows:

Arithmetic: $\frac{3}{8} \times \frac{9}{16} = \frac{3}{8} \times \frac{16}{9}$

$$= \frac{\cancel{3}}{\cancel{8}} \times \frac{\cancel{(8)}(2)}{\cancel{(3)}(3)} = \frac{2}{3}$$

Algebra: $\frac{x - 3y}{x + 3y} \div \frac{x^2 - 6xy + 9y^2}{x^2 + 7xy + 12y^2}$

$$= \frac{x - 3y}{x + 3y} \cdot \frac{x^2 + 7xy + 12y^2}{x^2 - 6xy + 9y^2}$$

$$= \frac{\cancel{x - 3y}}{\cancel{x + 3y}} \cdot \frac{\cancel{(x + 3y)}(x + 4y)}{\cancel{(x - 3y)}(x - 3y)}$$

$$= \frac{x + 4y}{x - 3y}$$

Practice problems. In the following problems, divide and reduce to lowest terms:

1. $\frac{x - 2}{x^2 + 4x + 4} \div \frac{1}{x^2 - 4}$

2. $\frac{2a - 1}{a^3 + 3a} \div \frac{a + 1}{a^2 + 3}$

3. $\frac{a^3 - 4a^2 + 3a}{a + 2} \div (a - 3)$

4. $\frac{6t + 12}{9t^2 + 6t - 24} \div \frac{8t - 12}{15t - 20}$

Answers:

1. $\frac{(x - 2)^2}{x + 2}$

2. $\frac{2a - 1}{a^2 + a}$

3. $\frac{a(a - 1)}{a + 2}$

4. $\frac{5}{4t - 6}$

ADDING AND SUBTRACTING FRACTIONS

The rules of arithmetic for adding and subtracting fractions are applicable to algebraic fractions. Fractions that are to be combined by addition or subtraction must have the same denominator. The numerators are then combined according to the operation indicated and the result is placed over the denominator. For example, in the expression

$$\frac{x - 4}{x - 2} + \frac{2 - 11x}{2 - x}$$

the second denominator will be the same as the first, if its sign is changed. The value of the fraction will remain the same if the sign of the numerator is also changed. Thus, we have the following simplification:

$$\frac{x - 4}{x - 2} + \frac{2 - 11x}{2 - x} = \frac{x - 4}{x - 2} + \frac{-(2 - 11x)}{-(2 - x)}$$

$$= \frac{x - 4}{x - 2} + \frac{11x - 2}{x - 2}$$

$$= \frac{x - 4 + 11x - 2}{x - 2}$$

$$= \frac{12x - 6}{x - 2}$$

$$= \frac{6(2x - 1)}{x - 2}$$

When the denominators are not the same, we must reduce all fractions to be added or subtracted to a common denominator and then proceed.

Consider, for example,

$$\frac{4}{x^2 - 4} + \frac{3}{x^2 - 4x - 12}$$

We first must find the least common denominator (LCD). Remember this is the least number that is exactly divisible by each of the denominators. To find such a number, as in arithmetic, we first separate each of the denominators into prime factors. The LCD will contain all of the various prime factors, each one as many times as it occurs in any of the denominators.

Factoring, we have

$$\frac{4}{(x + 2)(x - 2)} + \frac{3}{(x - 6)(x + 2)}$$

and the LCD is $(x + 2)(x - 2)(x - 6)$. Rewriting the fractions with this denominator and adding numerators, we have the following expression:

$$\frac{4(x - 6)}{(x + 2)(x - 2)(x - 6)} + \frac{3(x - 2)}{(x + 2)(x - 2)(x - 6)}$$

$$= \frac{4(x - 6) + 3(x - 2)}{\text{LCD}}$$

$$= \frac{4x - 24 + 3x - 6}{\text{LCD}}$$

$$= \frac{7x - 30}{(x + 2)(x - 2)(x - 6)}$$

As another example, consider

$$\frac{4}{x + 3} - \frac{x + 2}{x^2 + 4x + 3}$$

Factoring the denominator of the second fraction, we find that the LCD is $(x + 3)(x + 1)$. Rewriting the original fractions with the LCD as denominator, we may now combine the fractions as follows:

$$\frac{4(x + 1)}{(x + 3)(x + 1)} - \frac{(x + 2)}{(x + 3)(x + 1)}$$

$$= \frac{4x + 4 - x - 2}{(x + 3)(x + 1)}$$

$$= \frac{3x + 2}{(x + 3)(x + 1)}$$

Practice problems. Perform the indicated operations in each of the following problems:

1. $\frac{3x - 4}{x^2 + x - 2} - \frac{x - 2}{x - 1}$

2. $\frac{3a}{a^2 - 9} - \frac{3}{3 - a}$

3. $\frac{x - 3}{3x} + \frac{x + 2}{2x}$

4. $\frac{1}{a^4 - 1} - \frac{1}{a + 1}$

5. $\frac{3}{(a + 4)^2} - \frac{2}{a(a + 4)} + \frac{1}{6(a + 4)}$

Answers:

1. $\frac{3x - x^2}{(x + 2)(x - 1)}$

2. $\frac{6a + 9}{(a + 3)(a - 3)}$

3. $\frac{5}{6}$

4. $\frac{2 - a^3 + a^2 - a}{(a^2 + 1)(a + 1)(a - 1)}$

5. $\frac{a^2 + 10a - 48}{6a(a + 4)^2}$

CHAPTER 11

LINEAR EQUATIONS IN ONE VARIABLE

One of the principal reasons for an intensive study of polynomials, grouping symbols, factoring, and fractions is to prepare for solving equations. The equation is perhaps the most important tool in algebra, and the more skillful the student becomes in working with equations, the greater will be his ease in solving problems.

Before learning to solve equations, it is necessary to become familiar with the words used in the discussion of them. An EQUATION is a statement that two expressions are equal in value. Thus,

$$4 + 5 = 9$$

and

$$A = lw$$

(Area of a rectangle = length x width)

are equations. The part to the left of the equality sign is called the LEFT MEMBER, or first member, of the equation. The part to the right is the RIGHT MEMBER, or second member, of the equation.

The members of an equation are sometimes thought of as corresponding to two weights that balance a scale. (See fig. 11-1.) This comparison is often helpful to students who are learning to solve equations. It is obvious, in the case of the scale, that any change made in one pan must be accompanied by an equal change in the other pan. Otherwise the scale will not balance. Operations on equations are based on the same principle. The members must be kept balanced or the equality is lost.

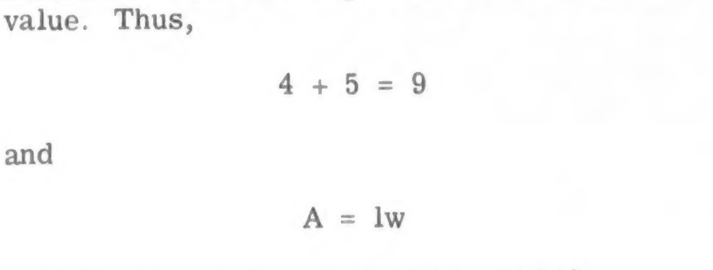


Figure 11-1. Equation compared to a balance scale.

CONSTANTS AND VARIABLES

Expressions in algebra consist of constants and variables. A CONSTANT is a quantity whose value remains the same throughout a particular problem. A VARIABLE is a quantity whose value is free to vary.

There are two kinds of constants--fixed and arbitrary. Numbers such as 7, -3, 1/2, and π are examples of FIXED constants. Their values never change. In $5x + 7 = 0$, the numbers 5 and 7 are fixed constants.

ARBITRARY constants can be assigned different values for different problems. Arbitrary constants are indicated by letters—quite often letters at the beginning of the alphabet such as a, b, c, and d. In

$$ax + b = 0,$$

the letters a and b represent arbitrary constants. The form $ax + b = 0$ represent many linear equations. If we give a and b particular values, say $a = 5$ and $b = 7$, then these constants become fixed, for this particular problem, and the equation becomes

$$5x + 7 = 0$$

A variable may have one value or it may have many values in a discussion. The letters at the end of the alphabet, such as x, y, z, and w, usually are used to represent variables. In $5x + 7$, the letter x is the variable. If $x = 1$, then

$$5x + 7 = 5 + 7 = 12$$

If $x = 2$, then

$$5x + 7 = 5(2) + 7 = 10 + 7 = 17$$

and so on for as many values of x as we desire to select.

If the expression 5x + 7 is set equal to some particular number, say -23, then the resulting equality

$$5x + 7 = -23$$

holds true for just one value of x. The value is -6, since

$$5(-6) + 7 = -23$$

In an algebraic expression, terms that contain a variable are called VARIABLE TERMS. Terms that do not contain a variable are CONSTANT TERMS. The expression 5x + 7 contains one variable term and one constant term. The variable term is 5x, while 7 is the constant term. In ax + b, ax is the variable term and b is the constant term.

A variable term often is designated by naming the variable it contains. In 5x + 7, 5x is the x-term. In ax + by, ax is the x-term, while by is the y-term.

DEGREE OF AN EQUATION

The degree of an equation that has not more than one variable in each term is the exponent of the highest power to which that variable is raised in the equation. The equation

$$3x - 17 = 0$$

is a FIRST-DEGREE equation, since x is raised only to the first power.

An example of a SECOND-DEGREE equation is

$$5x^2 - 2x + 1 = 0.$$

The equation,

$$4x^3 - 7x^2 = 0,$$

is of the THIRD DEGREE.

The equation,

$$3x - 2y = 5$$

is of the first degree in two variables, x and y. When more than one variable appears in a term, as in xy = 5, it is necessary to add the exponents of the variables within a term to get the degree of the equation. Since 1 + 1 = 2, the equation xy = 5 is of the second dgeree.

LINEAR EQUATIONS

Graphs are used in many different forms to give visual pictures of certain related facts. For example, they are used to show business trends, production output, continued individual attainment, and so forth. We find bar graphs, line graphs, circle graphs, and many other types, each of which is used for a particular need. In algebra, graphs are also used to give a visual picture containing a great deal of information about equations.

Sometimes many numerical values, when substituted for the variables of an equation, will satisfy the conditions of the equation. On a particular type of graph (which will be explained fully in chapter 12) several of these values are plotted (located), and when enough are plotted, a line is drawn through these points. For each particular equation a certain type of curve results. For equations in the first degree in one or two variables, the resulting shape of the "curve" is a straight line. Thus, the name LINEAR EQUATION is derived. Equations of a higher degree form various other shapes. The name "linear equation" now applies to equations of the first degree, regardless of the number of variables they contain. Chapter 12 shows how an equation may be pictured on a graph. The purpose and value of graphing an equation will also be developed.

IDENTITIES

If a statement of equality involves one or more variables, it may be either an IDENTITY (identical equation) or a CONDITIONAL EQUATION. An identity is an equality that states a fact, such as the following examples:

1. $9 + 5 = 14$
2. $2n + 5n = 7n$
3. $6(x - 3) = 6x - 18$

Notice that equation 3 merely shows the factored form of 6x - 18 and holds true when any value of x is substituted. For example, if x = 5, it becomes

$$6(5-3) = 6(5) - 18$$

$$6(2) = 30 - 18$$

$$12 = 12$$

If x assumes the negative value -10, this identity becomes

$$6(-10-3) = 6(-10)-18$$
$$6(-13) = -60-18$$
$$-78 = -78$$

An identity is established when both sides of the equality have been reduced to the same number or the same expression. When 5 is substituted for x, the value of either side of $6(x-3) = 6x - 18$ is 12. When -10 is substituted for x, the value on either side is -78. The fact that this equality is an identity can be shown also by factoring the right side so that the equality becomes

$$6(x-3) = 6(x-3)$$

The expressions on the two sides of the equality are identical.

CONDITIONAL EQUATIONS

A statement such as $2x-1 = 0$ is an equality only when x has one particular value. Such a statement is called a CONDITIONAL EQUATION, since it is true only under the condition that $x = 1/2$. Likewise, the equation $y - 7 = 8$ holds true only if $y = 15$.

The value of the variable for which an equation in one variable holds true is a ROOT, or SOLUTION, of the equation. When we speak of solving equations in algebra, we refer to conditional equations. The solution of a conditional equation can be verified by substituting for the variable its value, as determined by the solution.

The solution is correct if the equality reduces to an identity. For example, if 1/2 is substituted for x in $2x - 1 = 0$, the result is

$$2\left(\frac{1}{2}\right) - 1 = 0$$
$$1 - 1 = 0$$
$$0 = 0 \quad \text{(an identity)}$$

The identity is established for $x = \frac{1}{2}$, since the value of each side of the equality reduces to zero.

SOLVING LINEAR EQUATIONS

Solving a linear equation in one variable means finding the value of the variable that makes the equation true. For example, 11 is the SOLUTION of $x - 7 = 4$, since $11 - 7 = 4$. The number 11 is said to SATISFY the equation. Basically, the operation used in solving equations is to manipulate both members, by addition, subtraction, multiplication, or division until the value of the variable becomes apparent. This manipulation may be accomplished in a straightforward manner by use of the axioms outlined in chapter 3 of this course. These axioms may be summed up in the following rule: If both members of an equation are increased, decreased, multiplied, or divided by the same number, or by equal numbers, the results will be equal. (Division by zero is excluded.)

As mentioned earlier, an equation may be compared to a balance. What is done to one member must also be done to the other to maintain a balance. An equation must always be kept in balance or the equality is lost. We use the above rule to remove or adjust terms and coefficients until the value of the variable is discovered. Some examples of equations solved by means of the four operations mentioned in the rule are given in the following paragraphs.

ADDITION

Find the value of x in the equation

$$x - 3 = 12$$

As in any equation, we must isolate the variable on either the right or left side. In this problem, we leave the variable on the left and perform the following steps:

1. Add 3 to both members of the equation, as follows:

$$x - 3 + 3 = 12 + 3$$

In effect, we are "undoing" the subtraction indicated by the expression $x - 3$, for the purpose of isolating x in the left member.

2. Combining terms, we have

$$x = 15$$

SUBTRACTION

Find the value of x in the equation

$$x + 14 = 24$$

1. Subtract 14 from each member. In effect, this undoes the addition indicated in the expression $x + 14$.

$$x + 14 - 14 = 24 - 14$$

2. Combining terms, we have

$$x = 10$$

MULTIPLICATION

Find the value of y in the equation

$$\frac{y}{5} = 10$$

1. The only way to remove the 5 so that the y can be isolated is to undo the indicated division. Thus we use the inverse of division, which is multiplication. Multiplying both members by 5, we have the following:

$$5\left(\frac{y}{5}\right) = 5(10)$$

2. Performing the indicated multiplications, we have

$$y = 50$$

DIVISION

Find the value of x in the equation

$$3x = 15$$

1. The multiplier 3 may be removed from the x by dividing the left member by 3. This must be balanced by dividing the right member by 3 also, as follows:

$$\frac{3x}{3} = \frac{15}{3}$$

2. Performing the indicated divisions, we have

$$x = 5$$

Practice problems. Solve the following equations:

1. $m + 2 = 8$
2. $x - 5 = 11$
3. $6x = -48$
4. $\frac{x}{14} = 2$
5. $2n = 5$
6. $\frac{1}{6}y = 6$

Answers:

1. $m = 6$
2. $x = 16$
3. $x = -8$
4. $x = 28$
5. $n = 2\frac{1}{2}$
6. $y = 36$

SOLUTIONS REQUIRING MORE THAN ONE OPERATION

Most equations involve more steps in their solutions than the simple equations already described, but the basic operations remain unchanged. If the basic axioms are kept well in mind, these more complicated equations will not become too difficult. Equations may require one or all of the basic operations before a solution can be obtained.

Subtraction and Division

Find the value of x in the following equation:

$$2x + 4 = 16$$

1. The term containing x is isolated on the left by subtracting 4 from the left member. This operation must be balanced by also subtracting 4 from the right member, as follows:

$$2x + 4 - 4 = 16 - 4$$

2. Performing the indicated operations, we have

$$2x = 12$$

3. The multiplier 2 is removed from the x by dividing both sides of the equation by 2, as follows:

$$\frac{2x}{2} = \frac{12}{2}$$

$$x = 6$$

Addition, Multiplication, and Division

Find the value of y in the following equation:

$$\frac{3y}{2} - 4 = 11$$

1. Isolate the term containing y on the left by adding 4 to both sides, as follows:

$$\frac{3y}{2} - 4 + 4 = 11 + 4$$

$$\frac{3y}{2} = 15$$

2. Since the 2 will not divide the 3 exactly, multiply the left member by 2 in order to eliminate the fraction. This operation must be balanced by multiplying the right member by 2, as follows:

$$2\left(\frac{3y}{2}\right) = 2(15)$$

$$3y = 30$$

3. Divide both members by 3, in order to isolate the y in the left member, as follows:

$$\frac{3y}{3} = \frac{30}{3}$$

$$y = 10$$

Equations Having the Variable in More Than One Term

Find the value of x in the following equation:

$$\frac{3x}{4} + x = 12 - x$$

1. Rewrite the equation with no terms containing the variable in the right member. This requires adding x to the right member to eliminate the -x term, and balance requires that we also add x to the left member, as follows:

$$\frac{3x}{4} + x + x = 12 - x + x$$

$$\frac{3x}{4} + 2x = 12$$

2. Since the 4 will not divide the 3 exactly, it is necessary to multiply the first term by 4 to eliminate the fraction. However, notice that this multiplication cannot be performed on the first term only; any multiplier which is introduced for simplification purposes must be applied to the entire equation. Thus each term in the equation is multiplied by 4, as follows:

$$4\left(\frac{3x}{4}\right) + 4(2x) = 4(12)$$

$$3x + 8x = 48$$

3. Add the terms containing x and then divide both sides by 11 to isolate the x in the left member, as follows:

$$11x = 48$$

$$x = \frac{48}{11}$$

$$= 4\frac{4}{11}$$

Practice problems. Solve each of the following equations:

1. $x - 1 = \frac{1}{2}$
2. $\frac{y}{3} + y = 8$
3. $\frac{x}{4} + 3x = 7$
4. $4 - 7x = 9 - 8x$
5. $\frac{y}{2} + 6y = 13$
6. $\frac{1}{2}x - 2x = 25 + x$

Answers:

1. $x = 3/2$
2. $y = 6$
3. $x = 28/13$
4. $x = 5$
5. $y = 2$
6. $x = -10$

EQUATIONS WITH LITERAL COEFFICIENTS

As stated earlier, the first letters of the alphabet usually represent known quantities (constants), and the last letters represent unknown quantities (variables). Thus, we usually solve for x, y, or z.

An equation such as

$$ax - 8 = bx - 5$$

has letters as coefficients. Equations with literal coefficients are solved in the same way as equations with numerical coefficients, except that when an operation cannot actually be performed, it merely is indicated.

In solving for x in the equation

$$ax - 8 = bx - 5$$

subtract bx from both members and add 8 to both members. The result is

$$ax - bx = 8 - 5$$

Since the subtraction on the left side cannot actually be performed, it is indicated. The quantity, a - b, is the coefficient of x when terms are collected. The equation takes the form

$$(a-b)\ x = 3$$

Now divide both sides of the equation by a-b. Again the result can be only indicated. The solution of the equation is

$$x = \frac{3}{a-b}$$

In solving for y in the equation

$$ay + b = 4$$

subtract b from both members as follows:

$$ay = 4 - b$$

Dividing both members by a, the solution is

$$y = \frac{4-b}{a}$$

Practice problems. Solve for x in each of the following:

1. $3 + x = b$
2. $4x = 8 + t$
3. $3x + 6m = 7m$
4. $ax - 2(x + b) = 3a$

Answers:

1. $x = b - 3$
2. $x = \frac{8 + t}{4}$
3. $x = \frac{m}{3}$
4. $x = \frac{3a + 2b}{a - 2}$

REMOVING SIGNS OF GROUPING

If signs of grouping appear in an equation they should be removed in the manner indicated in chapter 9 of this course. For example, solve the equation

$$5 = 24 - [x-12(x-2) - 6(x-2)]$$

Notice that the same expression, x-2, occurs in both parentheses. By combining the terms containing (x-2), the equation becomes

$$5 = 24 - [x-18(x-2)]$$

Next, remove the parentheses and then the bracket, obtaining

$$\begin{aligned} 5 &= 24 - [x-18x + 36] \\ &= 24 - [36 - 17x] \\ &= 24 - 36 + 17x \\ &= -12 + 17x \end{aligned}$$

Subtracting 17x from both members and then subtracting 5 from both members, we have

$$\begin{aligned} -17x &= - 12 - 5 \\ -17x &= - 17 \end{aligned}$$

Divide both members by -17. The solution is

$$x = 1$$

EQUATIONS CONTAINING FRACTIONS

To solve for x in an equation such as

$$\frac{2x}{3} + \frac{x}{12} - 1 = \frac{1}{4} + \frac{x}{2}$$

first clear the equation of fractions. To do this, find the least common denominator of the fractions. Then multiply both sides of the equation by the LCD.

The least common denominator of 3, 12, 4, and 2 is 12. Multiply both sides of the equation by 12. The resulting equation is

$$8x + x - 12 = 3 + 6x$$

Subtract 6x from both members, add 12 to both members, and collect like terms as follows:

$$\begin{aligned} 9x - 6x &= 12 + 3 \\ 3x &= 15 \end{aligned}$$

The solution is

$$x = 5$$

To prove that $x = 5$ is the correct solution, substitute 5 for x in the original equation and show that both sides of the equation reduce to the same value. The result of substitution is

$$\frac{2(5)}{3} + \frac{5}{12} - 1 = \frac{1}{4} + \frac{5}{2}$$

In establishing an identity, the two sides of the equality are treated separately, and the operations are performed as indicated. Sometimes, as here, fractions occur on both sides of the equality, and it is desirable to find the least common denominator for more than one set of fractions. The same denominator could be used on both sides of the equality, but this might make some of the terms of the fractions larger than necessary.

Proceeding in establishing the identity for $x = 5$ in the foregoing equation we obtain

$$\frac{10}{3} + \frac{5}{12} - \frac{3}{3} = \frac{1}{4} + \frac{10}{4}$$

$$\frac{7}{3} + \frac{5}{12} = \frac{11}{4}$$

$$\frac{28}{12} + \frac{5}{12} = \frac{11}{4}$$

$$\frac{33}{12} = \frac{11}{4}$$

$$\frac{11}{4} = \frac{11}{4}$$

Each member of the equality has the value 11/4 when $x = 5$. The fact that the equation becomes an identity when x is replaced by 5 proves that $x = 5$ is the solution.

Practice problems. Solve each of the following equations:

1. $\frac{x}{4} - 2 = \frac{x}{6}$
2. $\frac{1}{2} - \frac{1}{v} = \frac{1}{3}$
3. $\frac{y}{2} - \frac{y}{3} = 5$
4. $\frac{3}{4x} = 6$

Answers:

1. $x = 24$
2. $v = 6$
3. $y = 30$
4. $x = 1/8$

GENERAL FORM OF A LINEAR EQUATION

The expression GENERAL FORM, in mathematics, implies a form to which all expressions or equations of a certain type can be reduced. The only possible terms in a linear equation in one variable are the first-degree term and the constant term. Therefore, the general form of a linear equation in one variable is

$$ax + b = 0$$

By selecting various values for a and b, this form can represent any linear equation in one variable after such an equation has been simplified. For example, if $a = 7$ and $b = 5$, $ax + b = 0$ represents the numerical equation

$$7x + 5 = 0$$

If $a = 2m - n$ and $b = p - q$, then $ax + b = 0$ represents the literal equation

$$(2m-n)x + p - q = 0$$

This equation is solved as follows:

$$(2m-n)x + (p-q) - (p-q) = 0 - (p-q)$$

$$(2m-n)x = 0 - (p-q)$$

$$x = \frac{-(p-q)}{2m-n} = \frac{q-p}{2m-n}$$

USING EQUATIONS TO SOLVE PROBLEMS

To solve a problem, we first translate the numerical sense of the problem into an equation. To see how this is accomplished, consider the following examples and their solutions.

EXAMPLE 1: Together Smith and Jones have \$120. Jones has 5 times as much as Smith. How much has Smith?

SOLUTION:

Step 1. Get the problem clearly in mind. There are two parts to each problem—what is given (the facts) and what we want to know (the question). In this problem we know that Jones has 5 times as much as Smith and together they have \$120. We want to know how much Smith has.

Step 2. Express the unknown as a letter. Usually we express the unknown or number we know the least about as a letter (conventionally we use x). Here we know the least about Smith's money. Let x represent the number of dollars Smith has.

Step 3. Express the other facts in terms of the unknown. If x is the number of dollars Smith has and Jones has 5 times as much, then 5x is the number of dollars Jones has.

Step 4. Express the facts as an equation. The problem will express or imply a relation between the expressions in steps 2 and 3. Smith's dollars plus Jones' dollars equal $120. Translating this statement into algebraic symbols, we have

$$x + 5x = 120$$

Solving the equation for x,

$$6x = 120$$
$$x = 20$$

Thus Smith has $20.

Step 5. Check: See if the solution satisfies the original statement of the problem. Smith and Jones have $120.

$20	+	$100	= $120
(Smith's money)		(Jones' money)	

EXAMPLE 2: Brown can do a piece of work in 5 hr. If Olsen can do it in 4 hr how long will it take them to do the work together?

SOLUTION:

Step 1. Given: Brown could do the work in 5 hr. Olsen could do it in 4 hours.

Unknown: How long it takes them to do the work together.

Step 2. Let x represent the time it takes them to do the work together.

Step 3. Then $\frac{1}{x}$ is the amount they do together in 1 hr. Also, in 1 hour Brown does $\frac{1}{5}$ of the work and Olsen does $\frac{1}{4}$ of the work.

Step 4. The amount done in 1 hr is equal to the part of the work done by Brown in 1 hr plus that done by Olsen in 1 hr.

$$\frac{1}{x} = \frac{1}{5} + \frac{1}{4}$$

Solving the equation,

$$20x\left(\frac{1}{x}\right) = 20x\left(\frac{1}{5}\right) + 20x\left(\frac{1}{4}\right)$$
$$20 = 4x + 5x$$
$$20 = 9x$$
$$\frac{20}{9} = x, \text{ or } x = 2\frac{2}{9} \text{ hours}$$

They complete the work together in $2\frac{2}{9}$ hours.

Step 5. Check: $2\frac{2}{9} \times \frac{1}{5}$ = amount Brown does

$2\frac{2}{9} \times \frac{1}{4}$ = amount Olsen does

$$\left(\frac{20}{9} \times \frac{1}{5}\right) + \left(\frac{20}{9} \times \frac{1}{4}\right) = \frac{4}{9} + \frac{5}{9} = \frac{9}{9}$$

Practice problems. Use a linear equation in one variable to solve each of the following problems:

1. Find three numbers such that the second is twice the first and the third is three times as large as the first. Their sum is 180.

2. A seaman drew $75.00 pay in dollar bills and five-dollar bills. The number of dollar bills was three more than the number of five-dollar bills. How many of each kind did he draw? (Hint: If x is the number of five-dollar bills, then 5x is the number of dollars they represent.)

3. Airman A can complete a maintenance task in 4 hr. Airman B requires only 3 hr to do the same work. If they work together, how long should it take them to complete the job?

Answers:

1. First number is 30.
   Second number is 60.
   Third number is 90.

2. Number of five-dollar bills is 12.
   Number of one-dollar bills is 15.

3. $1\frac{5}{7}$ hr.

INEQUALITIES

Modern mathematical thought gives considerable emphasis to the concept of inequality. A meaningful comparison between two quantities can be set up if they are related in some way, even though the relationship may not be one of equality.

The expression "number sentence" is often used to describe a general relationship which may be either an equality or an inequality. If the number sentence states an equality, it is an EQUATION; if it states an inequality, it is an INEQUATION.

ORDER PROPERTIES OF REAL NUMBERS

The idea of order, or relative rank according to size, is based upon two intuitive concepts: "greater than" and "less than." Mathematicians use the symbol $>$ to represent "greater than" and the symbol $<$ to represent "less than." For example, the inequation stating that 7 is greater than 5 is written in symbols as follows:

$$7 > 5$$

The inequation stating that x is less than 10 is written as follows:

$$x < 10$$

A "solution" of an inequation involving a variable is any number which may be substituted for the variable without changing the relationship between the left member and the right member. For example, the inequation $x < 10$ has many solutions. All negative numbers, and all positive numbers between 0 and 9, may be substituted for x successfully. These solutions comprise a set of numbers, called the SOLUTION SET.

The SENSE of an inequality refers to the direction in which the inequality symbol points. For example, the following two inequalities have opposite sense:

$$7 > 5$$
$$10 < 12$$

PROPERTIES OF INEQUALITIES

Inequations may be manipulated in accordance with specific operational rules, in a manner similar to that used with equations.

Addition

The rule for addition is as follows: If the same quantity is added to both members of an inequation, the result is an inequation having the same sense as the original inequation. The following examples illustrate this:

1.
$$5 < 8$$
$$5 + 2 < 8 + 2$$
$$7 < 10$$

The addition of 2 to both members does not change the sense of the inequation.

2.
$$5 < 8$$
$$5 + (-3) < 8 + (-3)$$
$$2 < 5$$

The addition of -3 to both members does not change the sense of the inequation.

Addition of the same quantity to both members is a useful method for solving inequations. In the following example, 2 is added to both members in order to isolate the x term on the left:

$$x - 2 > 6$$
$$x - 2 + 2 > 6 + 2$$
$$x > 8$$

Multiplication

The rule for multiplication is as follows: If both members of an inequation are multiplied by the same positive quantity, the sense of the resulting inequation is the same as that of the original inequation. This is illustrated as follows:

1.
$$-3 < -2$$
$$2(-3) < 2(-2)$$
$$-6 < -4$$

Multiplication of both members by 2 does not change the sense of th inequation.

2.
$$10 < 12$$
$$\frac{1}{2}(10) < \frac{1}{2}(12)$$
$$5 < 6$$

Multiplication of both members by 1/2 does not change the sense of the inequation.

Notice that example 2 illustrates division of both members by 2. Since any division can be rewritten as multiplication by a fraction, the multiplication rule is applicable to both multiplication and division.

Multiplication is used to simplify the solution of inequations such as the following:

$$\frac{x}{3} > 2$$

Multiply both members by 3:

$$3\left(\frac{x}{3}\right) > 3(2)$$

$$x > 6$$

Sense Reversal

If both sides of an inequation are multiplied or divided by the same negative number, the sense of the resulting inequation is reversed. This is illustrated as follows:

1. $$-4 < -2$$
$$(-3)(-4) > (-3)(-2)$$
$$12 > 6$$

2. $$7 > 5$$
$$(-2)(7) < (-2)(5)$$
$$-14 < -10$$

Sense reversal is useful in the solution of an inequation in which the variable is preceded by a negative sign, as follows:

$$2 - x < 4$$

Add -2 to both members to isolate the x term:

$$2 - x - 2 < 4 - 2$$

$$-x < 2$$

Multiply both members by -1:

$$x > -2$$

Practice problems. Solve each of the following inequations:

1. $x + 2 > 3$
2. $\frac{y}{3} - 1 < 2$
3. $3 - x < 6$
4. $4y > 8$

Answers:

1. $x > 1$
2. $y < 9$
3. $x > -3$
4. $y > 2$

GRAPHING INEQUALITIES

An inequation such as $x > 2$ can be graphed on a number line, as shown in figure 11-2.

The heavy line in figure 11-2 contains all values of x which comprise the solution set. Notice that this line continues indefinitely in the positive direction, as indicated by the arrow head. Notice also that the point representing $x = 2$ is designated by a circle. This signifies that the solution set does not contain the number 2.

Figure 11-3 is a graph of the inequation $x^2 > 4$. Since the square of any number greater than 2 is greater than 4, the solution set contains all values of x greater than 2. Furthermore, the solution set contains all values of x less than -2. This is because the square of any negative number smaller than -2 is a positive number greater than 4.

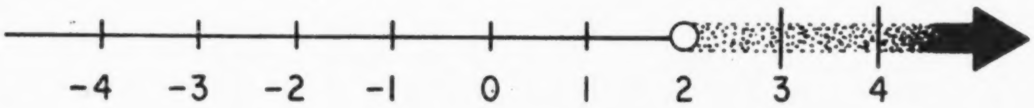


Figure 11-2.—Graph of the inequation $x > 2$.

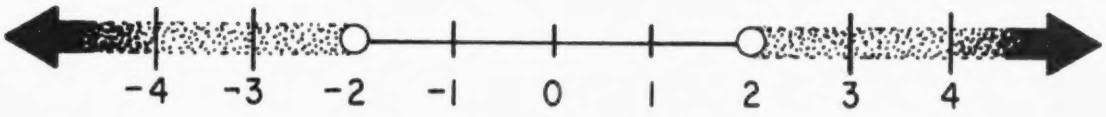


Figure 11-3.—Graph of $x^2 > 4$.

CHAPTER 12

LINEAR EQUATIONS IN TWO VARIABLES

Thus far in this course, discussions of equations have been limited to linear equations in one variable. Linear equations which have two variables are common, and their solution involves extending some of the procedures which have already been introduced.

RECTANGULAR COORDINATES

An outstanding characteristic of equations in two variables is their adaptability to graphical analysis. The rectangular coordinate system, which was introduced in chapter 3 of this course, is used in analyzing equations graphically. This system of vertical and horizontal lines, meeting each other at right angles and thus forming a rectangular grid, is often called the Cartesian coordinate system. It is named after the French philosopher and mathematician, Rene Descartes, who invented it.

COORDINATE AXES

The rectangular coordinate system is developed on a framework of reference similar to figure 3-2 in chapter 3 of this course. On a piece of graph paper, two lines are drawn intersecting each other at right angles, as in figure 12-1. The vertical line is usually labeled with the capital letter Y and called the Y axis. The horizontal line is usually labeled with the capital letter X and called the X axis. The point where the X and Y axes intersect is called the **ORIGIN** and is labeled with the letter o.

Above the origin, numbers measured along or parallel to the Y axis are positive; below the origin they are negative. To the right of the origin, numbers measured along or parallel to the X axis are positive; to the left they are negative.

COORDINATES

A point anywhere on the graph may be located by two numbers, one showing the distance of the point from the Y axis, and the other showing the distance of the point from the X axis.

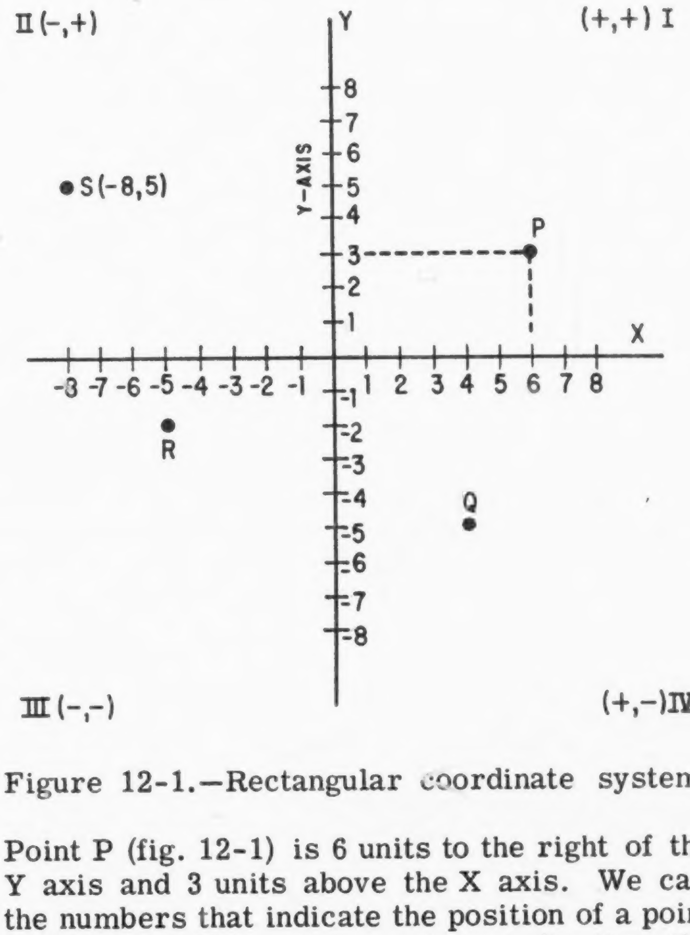


Figure 12-1.—Rectangular coordinate system.

Point P (fig. 12-1) is 6 units to the right of the Y axis and 3 units above the X axis. We call the numbers that indicate the position of a point COORDINATES. The number indicating the distance of the point measured horizontally from the origin is the X coordinate (6 in this example), and the number indicating the distance of the point measured vertically from the origin (3 in this example) is the Y coordinate.

In describing the location of a point by means of rectangular coordinates, it is customary to place the coordinates within parentheses and separate them with a comma. The X coordinate is always written first. The coordinates of point P (fig. 12-1) are written (6, 3). The coordinates for point Q are (4, -5); for point R, they are (-5, -2); and for point S, they are (-8, 5).

Usually when we indicate a point on a graph, we write a letter and the coordinates of the point. Thus, in figure 12-1, for point S, we write S(-8, 5). The other points would ordinarily be written, P(6, 3), Q(4, -5), and R(-5, -2). The Y coordinate of a point is often called its ORDINATE and the X coordinate is often called its ABSCISSA.

QUADRANTS

The X and Y axes divide the graph into four parts called QUADRANTS. In figure 12-1, point P is in quadrant I, point S is in quadrant II, R is in quadrant III, and Q is in quadrant IV. In the first and fourth quadrants, the X coordinate is positive, because it is to the right of the origin. In the second and third quadrant it is negative, because it is to the left of the origin. Likewise, the Y coordinate is positive in the first and second quadrants, being above the origin; it is negative in the third and fourth quadrants, being below the origin. Thus, we know in advance the signs of the coordinates of a point by knowing the quadrant in which the point appears. The signs of the coordinates in the four quadrants are shown in figure 12-1.

Locating points with respect to axes is called PLOTTING. As shown with point P (fig. 12-1), plotting a point is equivalent to completing a rectangle that has segments of the axes as two of its sides with lines dropped perpendicularly to the axes forming the other two sides. This is the reason for the name "rectangular coordinates."

PLOTTING A LINEAR EQUATION

A linear equation in two variables may have many solutions. For example, in solving the equation 2x - y = 5, we can find an unlimited number of values of x for which there will be a corresponding value of y. When x is 4, y is 3, since (2 x 4) - 3 = 5. When x is 3, y is 1, and when x is 6, y is 7. When we graph an equation, these pairs of values are considered coordinates of points on the graph. The graph of an equation is nothing more than a line joining the points located by the various pairs of numbers that satisfy the equation.

To picture an equation, we first find several pairs of values that satisfy the equation. For example, for the equation 2x - y = 5, we assign several values to x and solve for y. A convenient way to find values is to first solve the equation for either variable, as follows:

$$\begin{aligned} 2x - y &= 5 \\ -y &= -2x + 5 \\ y &= 2x - 5 \end{aligned}$$

Once this is accomplished, the value of y is readily apparent when values are substituted for x. The information derived may be recorded in a table such as table 12-1. We then lay off X and Y axes on graph paper, select some convenient unit distance for measurement along the axes, and then plot the pairs of values found for x and y as coordinates of points on the graph. Thus, we locate the pairs of values shown in table 12-1 on a graph, as shown in figure 12-2 (A).

Table 12-1.—Values of x and y in the equation 2x - y = 5.

If x = ------	-2	1	3	5	6	7	8
Then y = ---	-9	-3	1	5	7	9	11

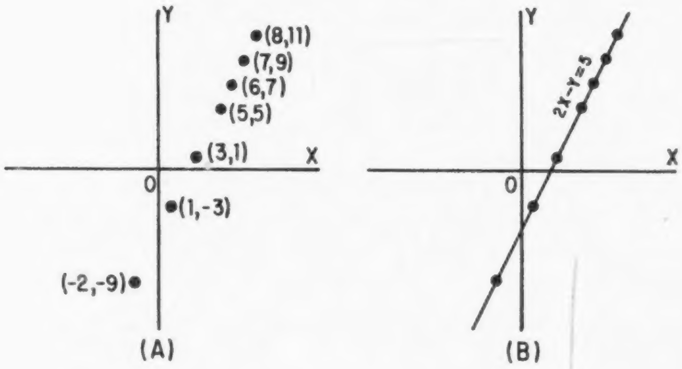


Figure 12-2.—Graph of 2x - y = 5.

Finally, we draw a line joining these points, as in figure 12-2 (B). It is seen that this is a straight line; hence the name "linear equation." Once the graph is drawn, it is customary to write the equation it represents along the line, as shown in figure 12-2 (B).

It can be shown that the graph of an equation is the geometric representation of all the points whose coordinates satisfy the conditions of the equation. The line represents an infinite number of pairs of coordinates for this equation. For example, selecting at random the point on the line where x is $2\frac{1}{2}$ and y is 0 and substituting these values in the equation, we find that they satisfy it. Thus,

$$2\left(2\frac{1}{2}\right) - 0 = 5$$

If two points that lie on a straight line can be located, the position of the line is known. The mathematical language for this is "Two points DETERMINE a straight line." We now know that the graph of a linear equation in two variables is a straight line. Since two points are sufficient to determine a straight line, a linear equation can be graphed by plotting two points and drawing a straight line through these points. Very often pairs of whole numbers which satisfy the equation can be found by inspection. Such points are easily plotted.

After the line is drawn through two points, it is well to plot a third point as a check. If this third point whose coordinates satisfy the equation lies on the line the graph is accurately drawn.

X AND Y INTERCEPTS

Any straight line which is not parallel to one of the axes has an X intercept and a Y intercept. These are the points at which the line crosses the X and Y axes. At the X intercept, the graph line is touching the X axis, and thus the Y value at that point is 0. At the Y intercept, the graph line is touching the Y axis; the X value at that point is 0.

In order to find the X intercept, we simply let y = 0 and find the corresponding value of x. The Y intercept is found by letting x = 0 and finding the corresponding value of y. For example, the line

$$5x + 3y = 15$$

crosses the Y axis at (0,5). This may be verified by letting x = 0 in the equation. The X intercept is (3,0), since x is 3 when y is 0. Figure 12-3 shows the line

$$5x + 3y = 15$$

graphed by means of the X and Y intercepts.

EQUATIONS IN ONE VARIABLE

An equation containing only one variable is easily graphed, since the line it represents lies parallel to an axis. For example, in

$$2y = 9$$

the value of y is

$$\frac{9}{2}, \text{ or } 4\frac{1}{2}$$

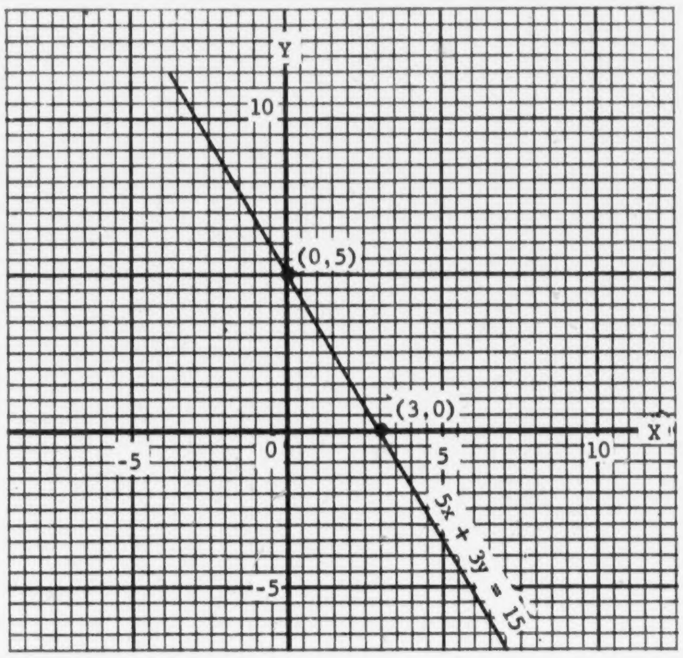


Figure 12-3.—Graph of 5x + 3y = 15.

The line 2y = 9 lies parallel to the X axis at a distance of $4\frac{1}{2}$ units above it. (See fig. 12-4.) Notice that each small division on the graph paper in figure 12-4 represents one-half unit.

The line 4x + 15 = 0 lies parallel to the Y axis. The value of x is $-\frac{15}{4}$. Since this value is negative, the line lies to the left of the Y axis at a distance of $3\frac{3}{4}$ units. (See fig. 12-4.)

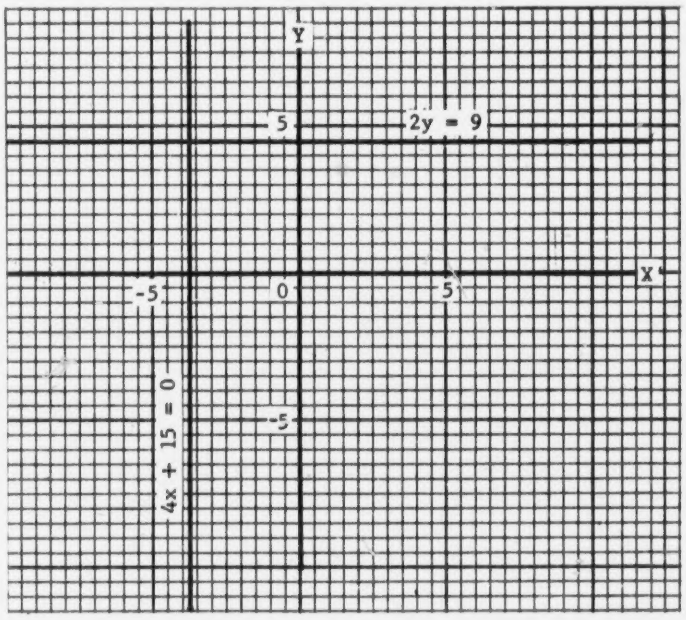


Figure 12-4.—Graphs of 2y = 9 and 4x + 15 = 0.

From the foregoing discussion, we arrive at two important conclusions:

1. A pair of numbers that satisfy an equation are the coordinates of a point on the graph of the equation.
2. The coordinates of any point on the graph of an equation will satisfy that equation.

SOLVING EQUATIONS IN TWO VARIABLES

A solution of a linear equation in two variables consists of a pair of numbers that satisfy the equation. For example, $x = 2$ and $y = 1$ constitute a solution of

$$3x - 5y = 1$$

When 2 is substituted for x and 1 is substituted for y, we have

$$3(2) - 5(1) = 1$$

The numbers $x = -3$ and $y = -2$ also form a solution. This is true because substituting -3 for x and -2 for y reduces the equation to an identity:

$$\begin{aligned} 3(-3) -5(-2) &= 1 \\ -9 + 10 &= 1 \\ 1 &= 1 \end{aligned}$$

Each pair of numbers (x, y) such as (2, 1) or (-3, -2) locates a point on the line $3x - 5y = 1$. Many more solutions could be found. Any two numbers that constitute a solution of the equation are the coordinates of a point on the line represented by the equation.

Suppose we were asked to solve a problem such as: Find two numbers such that their sum is 33 and their difference is 5. We could indicate the problem algebraically by letting x represent one number and y the other. Thus, the problem may be indicated by the two equations

$$\begin{aligned} x + y &= 33 \\ x - y &= 5 \end{aligned}$$

Considered separately, each of these equations represents a straight line on a graph. There are many pairs of values for x and y which satisfy the first equation, and many other pairs which satisfy the second equation. Our problem is to find ONE pair of values that will satisfy BOTH equations. Such a pair of values is said to satisfy both equations at the same time, or simultaneously. Hence, two equations for which we seek a common solution are called SIMULTANEOUS EQUATIONS. The two equations, taken together, comprise a SYSTEM of equations.

Graphical Solution

If there is a pair of numbers which may be substituted for x and y in two different equations, these numbers form the coordinates of a point which lies on the graph of each equation. The only way in which a point can lie on two lines simultaneously is for the point to be at the intersection of the lines. Therefore, the graphical solution of two simultaneous equations involves drawing their graphs and locating the point at which the graph lines intersect.

For example, when we graph the equations $x + y = 33$ and $x - y = 5$, as in figure 12-5, we see that they intersect in a single point. There is one pair of values comprising coordinates of that point (19, 14), and that pair of values satisfies both equations, as follows:

$$\begin{aligned} x + y &= 33 & \qquad x - y &= 5 \\ 19 + 14 &= 33 & \qquad 19 - 14 &= 5 \end{aligned}$$

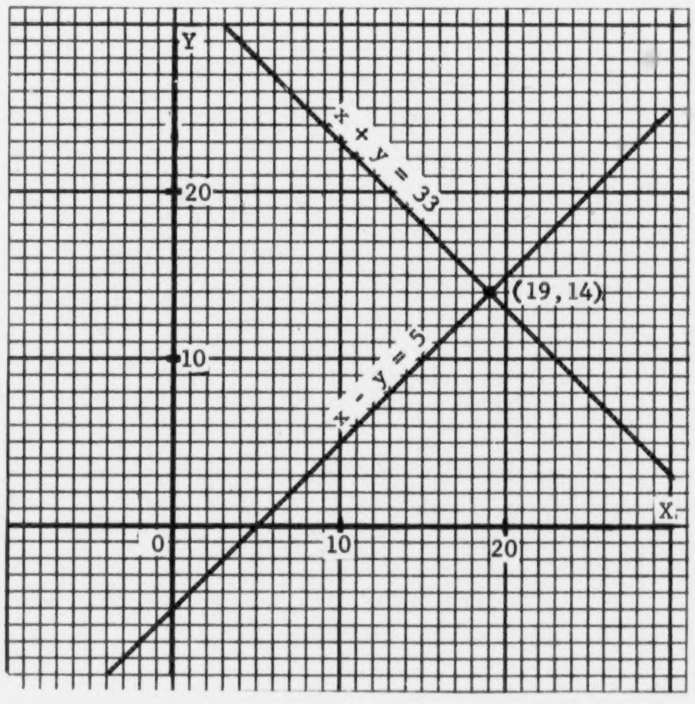


Figure 12-5.—Graph of $x + y = 33$ and $x = y = 5$.

This pair of numbers satisfies each equation. It is the only pair of numbers that satisfies the two equations simultaneously.

The graphical method is a quick and simple means of finding an approximate solution of two simultaneous equations. Each equation is graphed, and the point of intersection of the two lines is read as accurately as possible. A high degree of accuracy can be obtained but this, of course, is dependent on the precision with which the lines are graphed and the amount of accuracy possible in reading the graph. Sometimes the graphical method is quite adequate for the purpose of the problem.

Figure 12-6 shows the graphs of $x + y = 11$ and $x - y = -3$. The intersection appears to be the point (4, 7). Substituting $x = 4$ and $y = 7$ into the equations shows that this is the actual point of intersection, since this pair of numbers satisfies both equations.

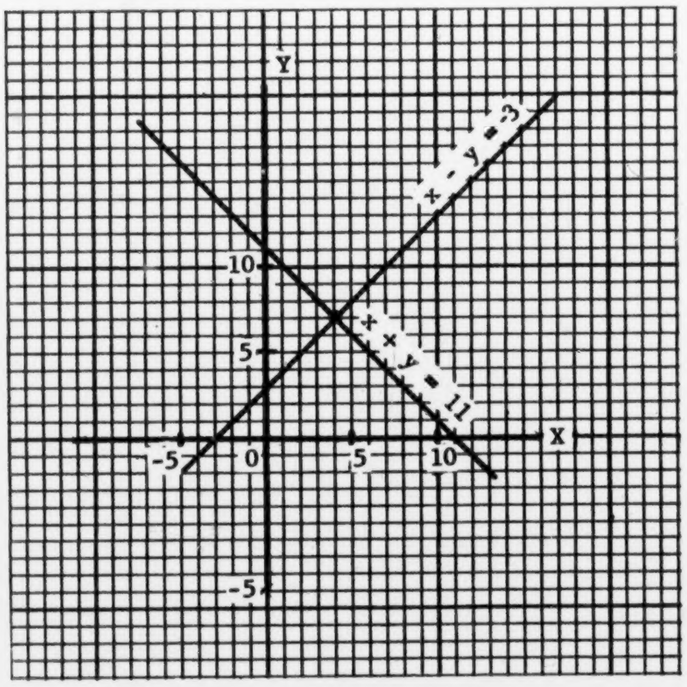


Figure 12-6.—Graph of $x + y = 11$ and $x - y = -3$.

The equations $7x - 8y = 2$ and $4x + 3y = 5$ are graphed in figure 12-7. The lines intersect where y is approximately 1/2 and x is approximately 5/6.

Practice problems. Solve the following simultaneous systems graphically:

1. $x + y = 8$
   $x - y = 2$

2. $3x + 2y = 12$
   $4x + 5y = 2$

Answers:

1. $x = 5$
   $y = 3$

2. $x = 8$
   $y = -6$

Addition Method

The addition method of solving systems of equations is illustrated in the following example:

$$\begin{array}{r} x - y = 2 \\ x + y = 8 \\ \hline 2x + 0 = 10 \\ x = 5 \end{array}$$

The result in the foregoing example is obtained by adding the left member of the first equation to the left member of the second, and adding the right member of the first equation to the right member of the second.

Having found the value of x, we substitute this value in either of the original equations to find the value of y, as follows:

$$\begin{aligned} x - y &= 2 \\ (5) - y &= 2 \\ -y &= 2 - 5 \\ -y &= -3 \\ y &= 3 \end{aligned}$$

Notice that the primary goal in the addition method is the elimination (temporarily) of one of the variables. If the coefficient of y is the same in both equations, except for its sign, adding the equations eliminates y as in the foregoing example. On the other hand, suppose that the coefficient of the variable which we desire to eliminate is exactly the same in both equations.

In the following example, the coefficient of x is the same in both equations, including its sign:

$$\begin{aligned} x + 2y &= 4 \\ x - 3y &= -1 \end{aligned}$$

Adding the equations would not eliminate either x or y. However, if we multiply both members of the second equation by -1, then addition will eliminate x, as follows:

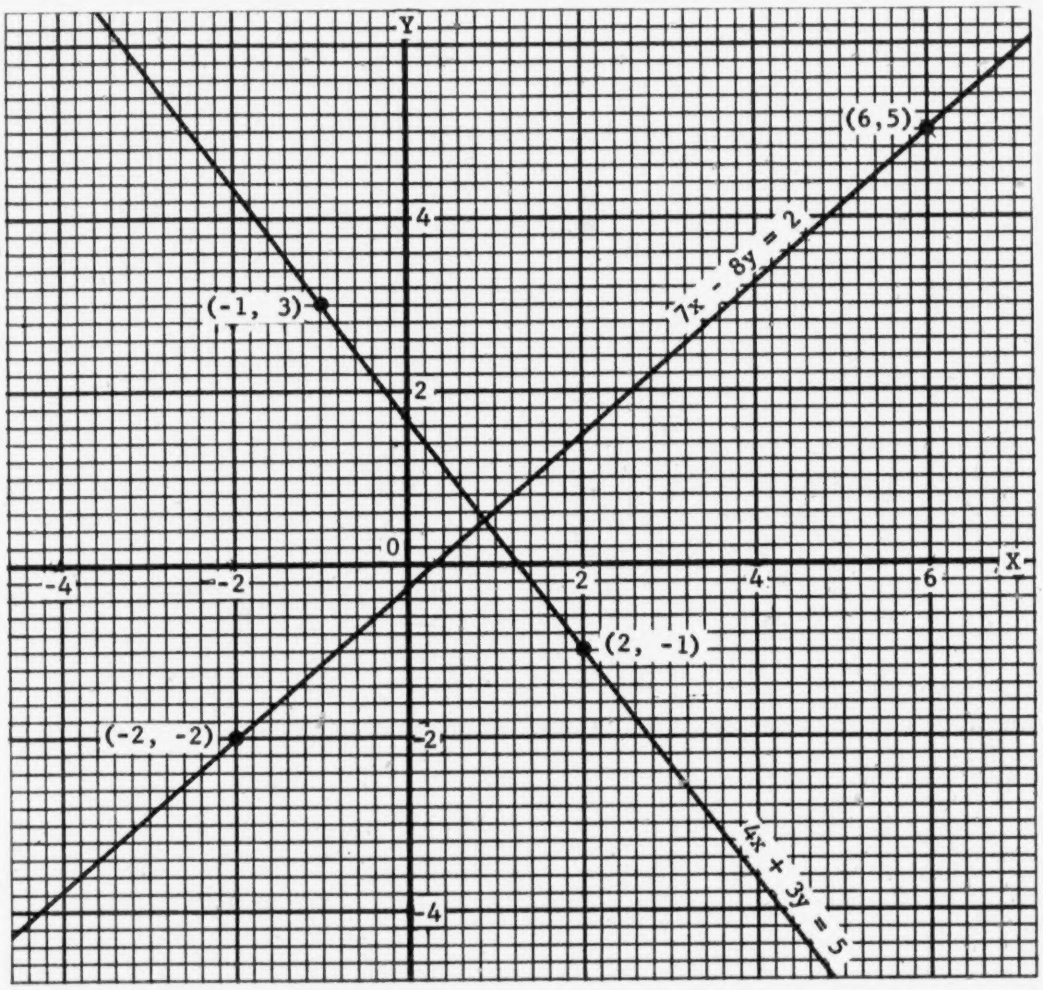


Figure 12-7.—Graph of 7x - 8y = 2 and 4x + 3Y = 5.

$$\begin{array}{r} x + 2y = 4 \\ -x + 3y = 1 \\ \hline 5y = 5 \\ y = 1 \end{array}$$

The value of x is found by substituting 1 for y in either of the original equations, as follows:

$$\begin{array}{r} x + 2(1) = 4 \\ x = 2 \end{array}$$

As a second example of the addition method, find the solution of the simultaneous equations

$$\begin{array}{r} 3x + 2y = 12 \\ 4x + 5y = 2 \end{array}$$

Here both x and y have unlike coefficients. The coefficients of one of the variables must be made the same, except for their signs.

The coefficients of x will be the same except for signs, if both members of the first equation are multiplied by 4 and both members of the second equation by -3. Then addition will eliminate x.

Following this procedure to get the value of y, we multiply the first equation by 4 and the second equation by -3, as follows:

$$\begin{array}{r} 12x + 8y = 48 \\ -12x - 15y = -6 \\ \hline -7y = 42 \\ y = -6 \end{array}$$

Substituting for y in the first equation to get the value of x, we have

$$\begin{array}{r} 3x + 2(-6) = 12 \\ x + 2(-2) = 4 \\ x - 4 = 4 \\ x = 8 \end{array}$$

This solution is checked algebraically by substituting 8 for x and -6 for y in each of the original equations, as follows:

1. $$3x + 2y = 12$$
$$3(8) + 2(-6) = 12$$
$$24 - 12 = 12$$

2. $$4x + 5y = 2$$
$$4(8) + 5(-6) = 2$$
$$32 - 30 = 2$$

Practice problems. Use the addition method to solve the following problems:

1. $x + y = 24$
   $x - y = 12$
2. $5t + 2v = 9$
   $3t - 2v = -5$
3. $x - 2y = -1$
   $2x + 3y = 12$
4. $2x + 7y = 3$
   $3x - 5y = 51$

Answers:

1. $x = 18$
   $y = 6$
2. $t = 1/2$
   $v = \frac{13}{4}$
3. $x = 3$
   $y = 2$
4. $x = 12$
   $y = -3$

Substitution Method

In some cases it is more convenient to use the substitution method of solving problems. In this method we solve one equation for one of the variables and substitute the value obtained into the other equation. This eliminates one of the variables, leaving an equation in one unknown. For example, find the solution of the following system:

$$4x + y = 11$$
$$x + 2y = 8$$

It is easy to solve for either y in the first equation or x in the second equation. Let us solve for y in the first equation. The result is

$$y = 11 - 4x$$

Since equals may be substituted for equals, we may substitute this value of y wherever y appears in the second equation. Thus,

$$x + 2(11 - 4x) = 8$$

We now have one equation that is linear in x; that is, the equation contains only the variable x.

Removing the parentheses and solving for x, we find that

$$x + 22 - 8x = 8$$
$$-7x = 8 - 22$$
$$-7x = -14$$
$$x = 2$$

To get the corresponding value of y, we substitute x = 2 in y = 11 - 4x. The result is

$$y = 11 - 4(2)$$
$$= 11 - 8$$
$$= 3$$

Thus, the solution for the two original equations is x = 2 and y = 3.

Practice problems. Solve the following systems by the substitution method:

1. $2x - 9y = 1$
   $x - 4y = 1$
2. $2x + y = 0$
   $2x - y = 1$
3. $5r + 2s = 23$
   $4r + s = 19$
4. $t - 4v = 1$
   $2t - 9v = 3$

Answers:

1. $x = 5$
   $y = 1$
2. $x = 1/4$
   $y = -1/2$
3. $r = 5$
   $s = -1$
4. $t = -3$
   $v = -1$

Literal Coefficients

Simultaneous equations with literal coefficients and literal constants may be solved for the value of the variables just as the other equations discussed in this chapter, with the exception that the solution will contain literal numbers. For example, find the solution of the system:

$$3x + 4y = a$$
$$4x + 3y = b$$

We proceed as with any other simultaneous linear equation. Using the addition method, we may proceed as follows: To eliminate the y term we multiply the first equation by 3 and the second equation by -4. The equations then become

$$\begin{array}{rl} 9x + 12y &= 3a \\ -16x - 12y &= -4b \\ \hline -7x \quad &= 3a - 4b \\ x \quad &= \frac{3a - 4b}{-7} \\ x &= \frac{4b - 3a}{7} \end{array}$$

To eliminate x, we multiply the first equation by 4 and the second equation by -3. The equations then become

$$\begin{array}{rl} 12x + 16y &= 4a \\ -12x - 9y &= -3b \\ \hline 7y &= 4a - 3b \\ y &= \frac{4a - 3b}{7} \end{array}$$

We may check in the same manner as that used for other equations, by substituting these values in the original equations.

INTERPRETING EQUATIONS

Recall that the general form for an equation in the first degree in one variable is $ax + b = 0$. The general form for first-degree equations in two variables is

$$ax + by + c = 0.$$

It is interesting and often useful to note what happens graphically when equations differ, in certain ways, from the general form. With this information, we know in advance certain facts concerning the equation in question.

LINES PARALLEL TO THE AXES

If in a linear equation the y term is missing, as in

$$2x - 15 = 0$$

the equation represents a line parallel to the Y axis and $7\frac{1}{2}$ units from it. Similarly, an equation such as

$$4y - 9 = 0$$

which has no x term, represents a line parallel to the X axis and $2\frac{1}{4}$ units from it. (See fig. 12-8.)

The fact that one of the two variables does not appear in an equation means that there are no limitations on the values the missing variable can assume. When a variable does not appear, it can assume any value from zero to plus or minus infinity. This can happen only if the line represented by the equation lies parallel to the axis of the missing variable.

Lines Passing Through the Origin

A linear equation, such as

$$4x + 3y = 0$$

that has no constant term, represents a line passing through the origin. This fact is obvious since $x = 0$, $y = 0$ satisfies any equation not having a constant term. (See fig. 12-8.)

Lines Parallel to Each Other

An equation such as

$$3x - 2y = 6$$

has all possible terms present. It represents a line that is not parallel to an axis and does not pass through the origin.

Equations that are exactly alike, except for the constant terms, represent parallel lines. As shown in figure 12-8, the lines represented by the equations

$$3x - 2y = -18 \text{ and } 3x - 2y = 6$$

are parallel.

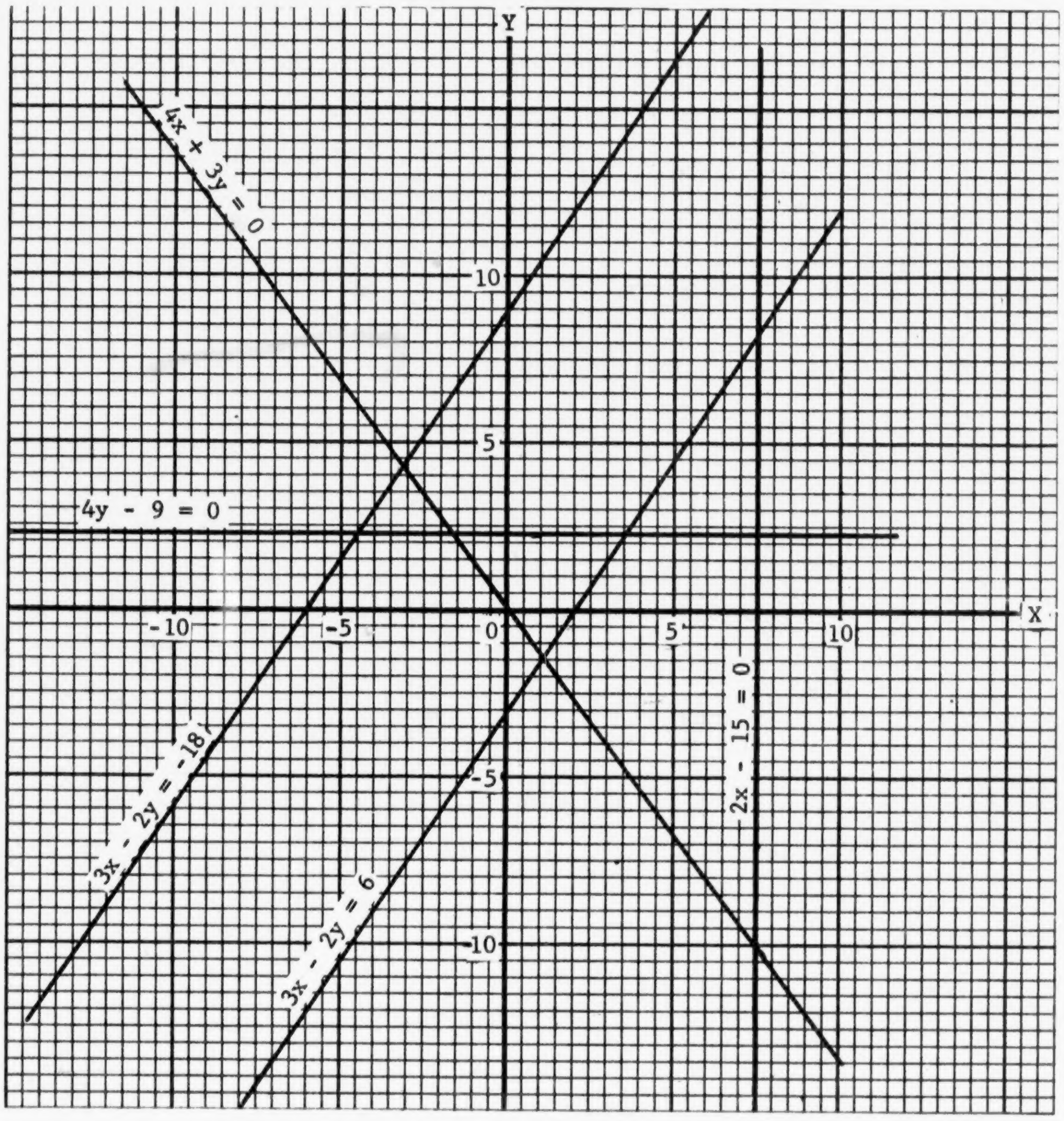


Figure 12-8.—Interpreting equations.

Parallel lines have the same slope. Changing the constant term moves a line away from or toward the origin while its various positions remain parallel to one another. Notice in figure 12-8 that the line $3x - 2y = 6$ lies closer to the origin than $3x - 2y = -18$. This is revealed at sight for any pair of lines by comparing their constant terms. That one which has the constant term of greater absolute value will lie farther from the origin. In this case $3x - 2y = -18$ will be farther from the origin since $|-18| > |6|$.

The fact that lines are parallel is indicated by the result when we try to solve two equations such as $3x - 2y = -18$ and $3x - 2y = 6$ simultaneously. Subtraction eliminates both x and y immediately. If both variables disappear, we cannot find values for them such that both equations are satisfied at the same time. This means tha there is no solution. No solution implies tha there is no point of intersection for the straigh lines represented by the equations. Lines tha do not intersect in the finite plane are parallel

USING TWO VARIABLES IN SOLVING WORD PROBLEMS

Many problems can be solved quickly an easily using one equation with one variable Other problems that might be rather difficult t solve in terms of one variable can easily b solved using two equations and two variables The difference in the two methods is shown i the following example, solved first by using on variable and then using two.

EXAMPLE: Find the two numbers such that half the first equals a third of the second and twice their sum exceeds three times the second by 4.

SOLUTION USING ONE VARIABLE:

1. Let x = the first number.
2. Then $\frac{x}{2} = \frac{1}{3}$ of the second number.
3. Thus $\frac{3x}{2}$ = the second number.

From the statement of the problem, we then have

$$2\left(x + \frac{3x}{2}\right) = 3\left(\frac{3x}{2}\right) + 4$$

$$2x + 3x = \frac{9x}{2} + 4$$

$$10x = 9x + 8$$

$$x = 8 \qquad \text{(first number)}$$

$$\frac{3x}{2} = 12 \qquad \text{(second number)}$$

SOLUTION USING TWO VARIABLES:

If we let x and y be the first and second numbers, respectively, we can write two equations almost directly from the statement of the problem. Thus,

1. $\frac{x}{2} = \frac{y}{3}$
2. $2(x + y) = 3y + 4$

Solving for x in the first equation and substituting this value in the second, we have

$$x = \frac{2y}{3}$$

$$2\left(\frac{2y}{3} + y\right) = 3y + 4$$

$$\frac{4y}{3} + 2y = 3y + 4$$

$$4y + 6y = 9y + 12$$

$$y = 12 \qquad \text{(second number)}$$

$$\frac{x}{2} = \frac{12}{3}$$

$$x = 8 \qquad \text{(first number)}$$

Thus, we see that the solution using two variables is more direct and simple. Often it would require a great deal of skill to manipulate a problem so that it might be solved using one variable; whereas the solution using two variables might be very simple. The use of two variables, of course, involves the fact that the student must be able to form two equations from the information given in the problem.

Practice problems. Solve the following problems using two variables:

1. A Navy tug averages 12 miles per hour downstream and 9 miles per hour upstream. How fast is the stream flowing?

2. The sum of the ages of two boys is 18. If 4 times the younger boy's age is subtracted from 3 times the older boy's age, the difference is 12. What are the ages of the two boys?

Answers:

1. $1\frac{1}{2}$ mph.
2. 6 years and 12 years.

INEQUALITIES IN TWO VARIABLES

Inequalities in two variables are of the following form:

$$x + y > 2$$

Many solutions of such an inequation are apparent immediately. For example, x could have the value 2 and y could have the value 3, since $2 + 3$ is greater than 2.

The existence of a large number of solutions suggests that a graph of the inequation would contain many points. The graph of an inequation in two unknowns is, in fact, an entire area rather than just a line.

PLOTTING ON THE COORDINATE SYSTEM

It would be extremely laborious to plot enough points at random to define an entire area of the coordinate system. Therefore our method consists of plotting a boundary line and shading the area, on one side of this line, wherein the solution points lie.

The equation of the boundary line is formed by changing the inequation to an equation. For example, the equation of the boundary line for the graph of

$$x + y > 2$$

is the equation

$$x + y = 2$$

Figure 12-9 is a graph of $x + y > 2$. Notice that the boundary line $x + y = 2$ is not solid. This is intended to indicate that points on the boundary line are not members of the solution set. Every point lying above and to the right of the boundary line is a member of the solution set. Any solution point may be verified by substituting its X and Y coordinates for x and y in the original inequation.

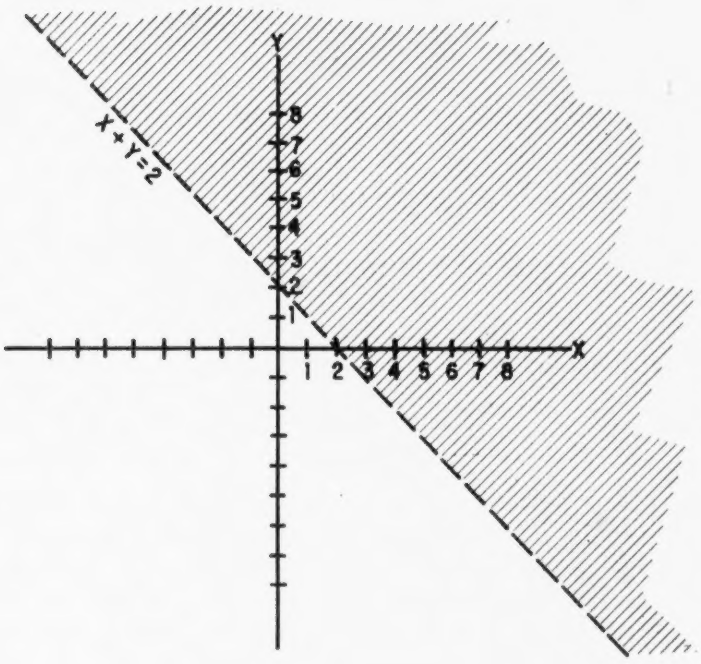


Figure 12-9.—Graph of $x + y > 2$.

SIMULTANEOUS INEQUALITIES

The areas representing the solutions of tw different inequations may overlap. If such a overlap occurs, the area of the overlap include all points whose coordinates satisfy both in equations simultaneously. An example of thi is shown in figure 12-10, in which the followin two inequations are graphed:

$$x + y > 2$$

$$x - y > 2$$

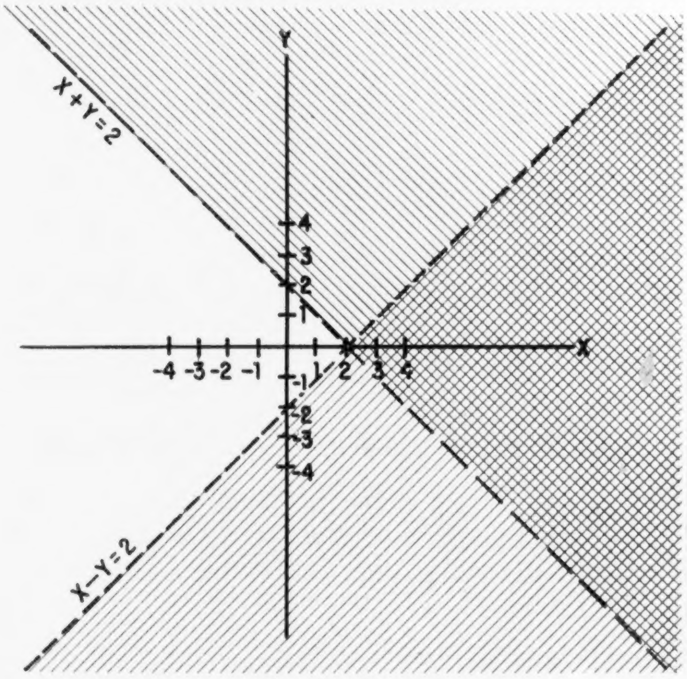


Figure 12-10.—Graph of $x + y > 2$ and $x - y > 2$

The double crosshatched area in figure 12-1 contains all points which comprise the solutio set for the system.

CHAPTER 13

RATIO, PROPORTION, AND VARIATION

The solution of problems based on ratio, proportion, and variation involves no new principles. However, familiarity with these topics will often lead to quick and simple solutions to problems that would otherwise be more complicated.

RATIO

The results of observation or measurement often must be compared with some standard value in order to have any meaning. For example, to say that a man can read 400 words per minute has little meaning as it stands. However, when his rate is compared to the 250 words per minute of the average reader, one can see that he reads considerably faster than the average reader. How much faster? To find out, his rate is divided by the average rate, as follows:

$$\frac{400}{250} = \frac{8}{5}$$

Thus, for every 5 words read by the average reader, this man reads 8. Another way of making this comparison is to say that he reads $1\frac{3}{5}$ times as fast as the average reader.

When the relationship between two numbers is shown in this way, they are compared as a RATIO. A ratio is a comparison of two like quantities. It is the quotient obtained by dividing the first number of a comparison by the second.

Comparisons may be stated in more than one way. For example, if one gear has 40 teeth and another has 10, one way of stating the comparison would be 40 teeth to 10 teeth. This comparison could be shown as a ratio in four ways as follows:

1. 40:10
2. $40 \div 10$
3. $\frac{40}{10}$
4. The ratio of 40 to 10.

When the emphasis is on "ratio," all of these expressions would be read, "the ratio of 40 to 10." The form $40 \div 10$ may also be read "40 divided by 10." The form $\frac{40}{10}$ may also be read "40 over 10."

Comparison by means of a ratio is limited to quantities of the same kind. For example, in order to express the ratio between 6 ft and 3 yd, both quantities must be written in terms of the same unit. Thus the proper form of this ratio is 2 yd : 3 yd, not 6 ft : 3 yd. When the parts of the ratio are expressed in terms of the same unit, the units cancel each other and the ratio consists simply of two numbers. In this example, the final form of the ratio is 2 : 3.

Since a ratio is also a fraction, all the rules that govern fractions may be used in working with ratios. Thus, the terms may be reduced, increased, simplified, and so forth, according to the rules for fractions. To reduce the ratio 15:20 to lowest terms, write the ratio as a fraction and then proceed as for fractions. Thus, 15:20 becomes

$$\frac{15}{20} = \frac{3}{4}$$

Hence the ratio of 15 to 20 is the same as the ratio of 3 to 4.

Notice the distinction in thought between $\frac{3}{4}$ as a fraction and $\frac{3}{4}$ as a ratio. As a fraction we think of $\frac{3}{4}$ as the single quantity "three-fourths." As a ratio, we think of $\frac{3}{4}$ as a comparison between the two numbers, 3 and 4. For example, the lengths of two sides of a triangle are $1\frac{9}{16}$ ft and 2 ft. To compare these lengths by means of a ratio, divide one number by the other and reduce to lowest terms, as follows:

$$\frac{1\frac{9}{16}}{2} = \frac{\frac{25}{16}}{2} = \frac{25}{32}$$

The two sides of the triangle compare as 25 to 32.

INVERSE RATIO

It is often desirable to compare the numbers of a ratio in the inverse order. To do this, we simply interchange the numerator and the denominator. Thus, the inverse of 15:20 is 20:15. When the terms of a ratio are interchanged, the INVERSE RATIO results.

Practice problems. In problems 1 through 6, write the ratio as a fraction and reduce to lowest terms. In problems 7 through 10, write the inverse of the given ratio.

1. The ratio of 5 lb to 15 lb
2. \$16 : \$12
3. $16 \div 4$
4. One quart to one gallon
5. 5x to 10x
6. $3\frac{1}{3} : 4\frac{1}{2}$
7. The ratio of 6 ft to 18 ft
8. $\frac{4}{8}$
9. 5 : 8
10. 15 to 21

Answers:

1. $\frac{1}{3}$
2. $\frac{4}{3}$
3. $\frac{4}{1}$
4. $\frac{1}{4}$
5. $\frac{1}{2}$
6. $\frac{20}{27}$
7. $\frac{3}{1}$
8. $\frac{2}{1}$
9. $\frac{8}{5}$
10. $\frac{7}{5}$

PROPORTION

Closely allied with the study of ratio is the subject of proportion. A PROPORTION is nothing more than an equation in which the members are ratios. In other words when two ratios are set equal to each other, a proportion is formed. The proportion may be written in three different ways as in the following examples:

$$15:20 :: 3:4$$

$$15:20 = 3:4$$

$$\frac{15}{20} = \frac{3}{4}$$

The last two forms are the most common. All these forms are read, "15 is to 20 as 3 is to 4." In other words, 15 has the same ratio to 20 as 3 has to 4.

One reason for the extreme importance of proportions is that if any three of the terms are given, the fourth may be found by solving a simple equation. In science many chemical and physical relations are expressed as proportions. Consequently, a familiarity with proportions will provide one method for solving many applied problems. It is evident from the last form shown, $\frac{15}{20} = \frac{3}{4}$, that a proportion is really a fractional equation. Therefore, all the rules for fraction equations apply.

TERMS OF A PROPORTION

Certain names have been given to the terms of the two ratios that make up a proportion. In a proportion such as 3:8 = 9:24, the first and the last terms (the outside terms) are called the EXTREMES. In other words, the numerator of the first ratio and the denominator of the second are called the extremes. The second and third terms (the inside terms) are called the MEANS. The means are the denominator of the first ratio and the numerator of the second. In the example just given, the extremes are 3 and 24; the means are 8 and 9.

Four numbers, such as 5, 8, 15, and 24, form a proportion if the ratio of the first two in the order named equals the ratio of the second two. When these numbers are set up as ratios with the equality sign between them, the members will reduce to an identity if a true proportion exists. For example, consider the following proportion:

$$\frac{5}{8} = \frac{15}{24}$$

In this proportion, $\frac{15}{24}$ must reduce to $\frac{5}{8}$ for the proportion to be true. Removing the same factor from both members of $\frac{15}{24}$ we have

$$\frac{5}{8} = \frac{3(5)}{3(8)}$$

The number 3 is the common factor that must be removed from both the numerator and the denominator of one fraction in order to show that the expression

$$\frac{5}{8} = \frac{15}{24}$$

is a true proportion. To say this another way, it is the factor by which both terms of the ratio $\frac{5}{8}$ must be multiplied in order to show that this ratio is the same as $\frac{15}{24}$.

Practice problems. For each of the following proportions, write the means, the extremes, and the factor of proportionality.

1. $\frac{3}{16} = \frac{15}{80}$
2. $4:5 = 12:15$
3. $\frac{25}{75} = \frac{1}{3}$
4. $12:3 :: 4:1$

Answers:

1. Means: 16 and 15

   Extremes: 3 and 80

   Factor of proportionality: 5

2. M: 5 and 12

   E: 4 and 15

   FP: 3

3. M: 75 and 1

   E: 25 and 3

   FP: 25

4. M: 3 and 4

   E: 12 and 1

   FP: 3

OPERATIONS OF PROPORTIONS

It is often advantageous to change the form of a proportion. There are rules for changing or combining the terms of a proportion without altering the equality between the members. These rules are simplifications of fundamental rules for equations; they are not new, but are simply adaptations of laws or equations presented earlier in this course.

Rule 1. In any proportion, the product of the means equals the product of the extremes.

This is perhaps the most commonly used rule of proportions. It provides a simple way to rearrange a proportion so that no fractions are present. In algebraic language the rule is illustrated as follows:

$$\frac{a}{b} = \frac{c}{d}$$

$$bc = ad$$

To prove this rule, we note that the LCD of the two ratios $\frac{a}{b}$ and $\frac{c}{d}$ is bd. Multiplying both members of the equation in its original form by this LCD, we have

$$bd \cdot \frac{a}{b} = bd \cdot \frac{c}{d}$$

$$ad = bc$$

The following numerical example illustrates the simplicity of rule 1:

$$\frac{3}{8} = \frac{9}{24}$$

$$8(9) = 3(24)$$

If one of the terms of a proportion is a variable to the first power as in

$$7:5 = x:6$$

the proportion is really a linear equation in one variable. Such an equation can be solved for the unknown.

Equating the products of the means and extremes produces the following:

$$5x = 42$$

$$x = 8\frac{2}{5}$$

Mean Proportional

When the two means of a proportion are the same quantity, that quantity is called the MEAN

PROPORTIONAL between the other two terms. In the proportion

$$\frac{a}{x} = \frac{x}{c}$$

x is the mean proportional between a and c.

Rule 2. The mean proportional between two quantities is the square root of their product. This rule is stated algebraically as follows:

$$\frac{a}{x} = \frac{x}{c}$$

$$x = \pm\sqrt{ac}$$

To prove rule 2, we restate the proportion and apply rule 1, as follows:

$$\frac{a}{x} = \frac{x}{c}$$

$$x^2 = ac$$

$$x = \pm\sqrt{ac}$$

Rule 2 is illustrated by the following numerical example:

$$\frac{2}{8} = \frac{8}{32}$$

$$8 = \sqrt{2(32)}$$

$$8 = \sqrt{64}$$

OTHER FORMS FOR PROPORTIONS

If four numbers, for example, a, b, c, and d, form a proportion, such as

$$\frac{a}{b} = \frac{c}{d}$$

they also form a proportion according to other arrangements.

Inversion

The four selected numbers are in proportion by INVERSION in the form

$$\frac{b}{a} = \frac{d}{c}$$

The inversion relationship is proved as follows, by first multiplying both members of the original proportion by $\frac{bd}{ac}$:

$$\left(\frac{bd}{ac}\right)\left(\frac{a}{b}\right) = \left(\frac{bd}{ac}\right)\left(\frac{c}{d}\right)$$

$$\frac{d}{c} = \frac{b}{a}$$

Note that the product of the means and the product of the extremes still yield the same equality as in the original proportion.

The inversion relationship may be illustrated by the following numerical example:

$$\frac{5}{8} = \frac{10}{16}$$

Therefore,

$$\frac{8}{5} = \frac{16}{10}$$

Alternation

The four selected numbers (a, b, c, and d) are in proportion by ALTERNATION in the following form:

$$\frac{a}{c} = \frac{b}{d}$$

To prove the alternation relationship, first multiply both sides of the original proportion by $\frac{b}{c}$, as follows:

$$\frac{a}{b} = \frac{c}{d}$$

$$\frac{b}{c}\left(\frac{a}{b}\right) = \frac{b}{c}\left(\frac{c}{d}\right)$$

$$\frac{a}{c} = \frac{b}{d}$$

The following numerical example illustrates alternation:

$$\frac{5}{8} = \frac{10}{16}$$

Therefore,

$$\frac{5}{10} = \frac{8}{16}$$

SOLVING PROBLEMS BY MEANS OF PROPORTION

One of the most common types of problems based on proportions involves triangles with roportional sides. Suppose that the corre-ponding sides of two triangles are known to be roportional. (See fig. 13-1.) The lengths of he sides of one triangle are 8, 9, and 11. The ength of the side of the second triangle corre-ponding to side 8 in the first triangle is 10. Ve wish to find the lengths of the remaining ides, b and c.

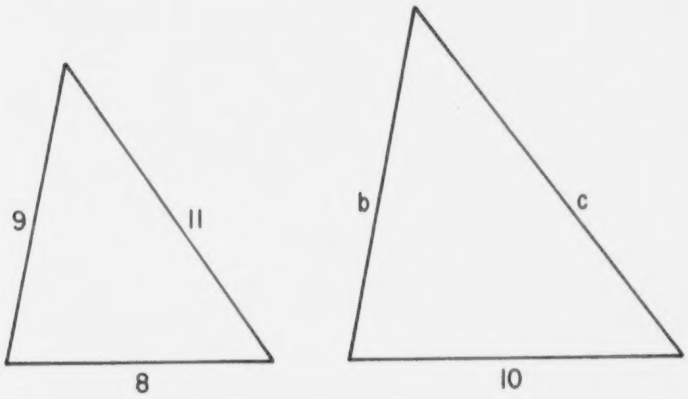


Figure 13-1.—Triangles with opposite sides proportional.

Since the corresponding sides are propor-tional, the pairs of corresponding sides may be used to form proportions as follows:

$$\frac{8}{10} = \frac{9}{b}$$

$$\frac{9}{b} = \frac{11}{c}$$

$$\frac{8}{10} = \frac{11}{c}$$

To solve for b, we use the proportion

$$\frac{8}{10} = \frac{9}{b}$$

and obtain the following result:

$$8b = 90$$

$$4b = 45$$

$$b = 11\frac{1}{4}$$

The solution for c is similar to that for b, using the proportion

$$\frac{8}{10} = \frac{11}{c}$$

with the following result:

$$8c = 110$$

$$c = 13\frac{3}{4}$$

The sides of the second triangle are 10, $11\frac{1}{4}$, and $13\frac{3}{4}$. The result can also be obtained by using the factor or proportionality. Since 8 and 10 are lengths of corresponding sides, we can write

$$8k = 10$$

$$k = \frac{10}{8} = \frac{5}{4}$$

The factor of proportionality is thus found to be $\frac{5}{4}$.

Multiplying any side of the first triangle by $\frac{5}{4}$ gives the corresponding side of the second triangle, as follows:

$$b = 9\left(\frac{5}{4}\right) = \frac{45}{4} = 11\frac{1}{4}$$

$$c = 11\left(\frac{5}{4}\right) = \frac{55}{4} = 13\frac{3}{4}$$

Proportional sides of similar triangles may be used to determine the height of an object by measuring its shadow. (See fig. 13-2.)

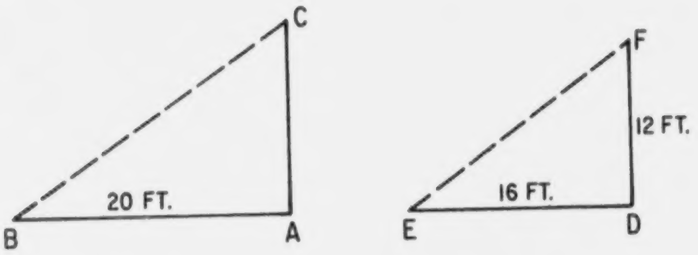


Figure 13-2.—Measuring height by shadow length.

In figure 13-2, mast AC casts a shadow 20 ft long when the shadow of DF is 16 ft long. Assuming that both masts are vertical and on level ground, triangle ABC is similar to triangle DEF and their corresponding sides are therefore proportional. Thus the height of AC may be found as follows:

$$\frac{AC}{12} = \frac{20}{16}$$

$$AC = \frac{(12)(20)}{16} = 15$$

Practice problems. In each of the following problems, set up a proportion and then solve for the unknown quantity:

1. Referring to figure 13-1, if the shortest side of the larger triangle is 16 units long, rather than 10, how long is side c?

2. If a mast 8 ft high casts a shadow 10 ft long, how high is a mast that casts a shadow 40 ft long?

Answers:

1. $\frac{8}{16} = \frac{11}{c}$

$8c = (11)(16)$

$c = \frac{(11)(16)}{8}$

$c = 22$

2. $\frac{8}{10} = \frac{h}{40}$

$\frac{(8)(40)}{10} = h$

$h = 32$

Word Problems

A knowledge of proportions often provides a quick method of solving word problems. The following problem is a typical example of the types that lend themselves to solution by means of proportion.

If an automobile runs 36 mi on 2 gal of gas, how many miles will it run on 12 gal? Comparing miles to miles and gallons to gallons, we have

$$36:x = 2:12$$

Rewriting this in fraction form, the solution is as follows:

$$\frac{36}{x} = \frac{2}{12}$$

$$2x = 12(36)$$

$$x = 6(36)$$

$$= 216 \text{ mi}$$

Practice problems. In each of the following problems, first set up a proportion and then solve for the unknown quantity:

1. The ratio of the speed of one aircraft to that of another is 2 to 5. If the slower aircraft has a speed of 300 knots, what is the speed of the faster aircraft?

2. If 6 seamen can empty 2 cargo spaces in 1 day, how many spaces can 150 seamen empty in 1 day?

3. On a map having a scale of 1 in. to 50 mi, how many inches represent 540 mi?

Answers:

1. 750 kt

2. 50

3. 10.8 in.

VARIATION

When two quantities are interdependent, changes in the value of one may have a predictable effect on the value of the other. Variation is the name given to the study of the effects of changes among related quantities. The three types of variation which occur frequently in the study of scientific phenomena are DIRECT, INVERSE, and JOINT.

DIRECT VARIATION

An example of direct variation is found in the following statement: The perimeter (sum of the lengths of the sides) of a square increases if the length of a side increases. In everyday language, this statement might become: The longer the side, the bigger the square. In mathematical symbols, using p for perimeter and s for the length of the side, the relationship is stated as follows:

$$p = 4s$$

Since the number 4 is constant, any variations which occur are the results of changes in p and s. Any increase or decrease in the size of s results in a corresponding increase or decrease in the size of p. Thus p varies in the same way (increasing or decreasing) as s. This explains the terminology which is frequently used: p varies directly as s.

In general, if a quantity can be expressed in terms of a second quantity multiplied by a constant, it is said to VARY DIRECTLY AS the second quantity. For example if x and y are variables and k is a constant, x varies directly as y, if $x = ky$. Thus, as y increases x increases, nd as y decreases, x decreases. There is a irect effect on x caused by any change in y.

The fact that x varies as y is sometimes inicated by $x \propto y$, or $x \sim y$. However, it is usully written in the form $x = ky$.

The relationship $x = ky$ is equivalent to $= k$. If one quantity varies directly as a second quantity, the ratio of the first quantity to he second quantity is a constant. Thus, whatver the value of x, where it is divided by y, he result will always be the same value, k.

A quantity that varies directly as another uantity is also said to be DIRECTLY PROORTIONAL to the second quantity. In $x = ky$, he coefficient of x is 1. The relationship $x = ky$ an be written in proportion form as

$$\frac{x}{k} = \frac{y}{1}$$

r

$$\frac{k}{x} = \frac{1}{y}$$

Notice that the variables, x and y, appear either in the numerators or in the denominators of the equal ratios. This implies that x and y are directly proportional. The constant, k, is the CONSTANT OF PROPORTIONALITY.

Practice problems. Write an equation showing the stated relationship, in each of the following problems:

1. The cost, C of a dozen wrenches varies directly as the price, p, of one wrench.

2. X is directly proportional to Y (use k as the constant of proportionality).

3. The circumference, C, of a circle varies directly as its diameter, d (use π as the constant of proportionality).

In the following problems, based on the formula $p = 4s$, find the appropriate word or symbol to fill the blank.

4. When s is doubled, p will be ____________.

5. When s is halved, p will be ____________.

6. ____________ is directly proportional to s.

Answers:

1. $C = 12p$
2. $X = kY$
3. $C = \pi d$
4. doubled
5. halved
6. p

Variation as the Power of a Quantity

Another form of direct variation occurs when a quantity varies as some power of another. For example, consider the formula

$$A = \pi r^2$$

Table 13-1 shows the values of r and the corresponding values of A.

Table 13-1.—Relation between values of radius and area in a circle.

When r = ----	1	2	3	4	5	7	9
Then A = ----	π	4π	9π	16π	25π	49π	81π

Notice how A changes as a result of a change in r. When r changes from 1 to 2, A changes from π to 4 times π or 2^2 times π. Likewise when r changes from 3 to 4, A changes not as r, but as the SQUARE of r. In general, one quantity varies as the power of another if it is equal to a constant times that quantity raised to the power. Thus, in an equation such as $x = ky^n$, x varies directly as the n^{th} power of y. As y increases, x increases but more rapidly than y, and as y decreases, x decreases, but again more rapidly.

Practice problems.

1. In the formula $V = e^3$, how does V vary?

2. In the formula $A = s^2$, if s is doubled how much is A increased?

3. In the formula $s = \frac{gt^2}{2}$, g is a constant. If t is halved, what is the resulting change in s?

Answers:

1. Directly as the cube of e.
2. It is multiplied by 4.
3. It is multiplied by $\frac{1}{4}$.

INVERSE VARIATION

A quantity VARIES INVERSELY as another quantity if the product of the two quantities is a constant. For example, if x and y are variables and k is a constant, the fact that x varies inversely as y is expressed by

$$xy = k$$

or

$$x = \frac{k}{y}$$

If values are substituted for x and y, we see that as one increases, the other must decrease, and vice versa. Otherwise, their product will not equal the same constant each time.

If a quantity varies inversely as a second quantity, it is INVERSELY PROPORTIONAL to the second quantity. In $xy = k$, the coefficient of k is 1. The equality $xy = k$ can be written in the form

$$\frac{x}{k} = \frac{1}{y}$$

or

$$\frac{k}{x} = \frac{y}{1}$$

Notice that when one of the variables, x or y, occurs in the numerator of a ratio, the other variable occurs in the denominator of the second ratio. This implies that x and y are inversely proportional.

Inverse variation may be illustrated by means of the formula for area of a rectangle. If A stands for area, L for length, and W for width, the expression for the area of a rectangle in terms of the length and width is

$$A = LW$$

Suppose that several rectangles, all having the same area but varying lengths and widths, are to be compared. Then $LW = A$ has the same form as $xy = k$, where A and k are constants. Thus L is inversely proportional to W, and W is inversely proportional to L.

If the constant area is 12 sq ft, this relationship becomes

$$LW = 12$$

If the length is 4 ft, the width is found as follows:

$$W = \frac{12}{L} = \frac{12}{4} = 3 \text{ ft}$$

If the length increases to 6 ft, the width decreases as follows:

$$W = \frac{12}{6} = 2 \text{ ft}$$

If a constant area is 12, the width of a rectangle decreases from 3 to 2 as the length increases from 4 to 6. When two inversely proportional quantities vary, one decreases as the other increases.

Another example of inverse variation is found in the study of electricity. The current flowing in an electrical circuit at a constant potential varies inversely as the resistance of the circuit. Suppose that the current, I, is 10 amperes when the resistance, R, is 11 ohms and it is desired to find the current when the resistance is 5 ohms.

Since I and R vary inversely, the equation for the relationship is $IR = k$, where k is the constant voltage. Therefore, $(10)(11) = k$. Also, when the resistance changes to 5 ohms, $(5)(I) = k$. Quantities equal to the same quantity are equal to each other, so we have the following equation:

$$5I = (10)(11)$$

$$I = \frac{110}{5} = 22$$

The current is 22 amperes when the resistance is 5 ohms. As the resistance decreases from 11 to 5 ohms, the current increases from 10 to 22 amperes.

One type of variation problem which tends to be confusing to the beginner involves rates of speed or rates of doing work. For example, if 7 men can complete a job in 20 days, how long will 50 men require to complete the same job? The strictly mechanical approach to this problem might result in the following false solution, relating men to men and days to days:

$$\frac{7 \text{ men}}{50 \text{ men}} = \frac{20 \text{ days}}{T}$$

However, a little thought brings out the fact that we are dealing with an INVERSE relationship rather than a direct one. In other words, the more men we have, the less time is required. Therefore, the correct solution requires that we use an inverse proportion; that is, we must invert one of the ratios as follows:

$$\frac{7}{50} = \frac{T}{20}$$

$$T = \frac{(7)(20)}{50} = 2\frac{4}{5} \text{ days}$$

Practice problems. In problems 1 and 2, express the given data as a proportion, using k as the constant of proportionality.

1. The rate, r, at which a vessel travels in going a certain distance varies inversely as the time, t.

2. The volume, V, of a gas varies inversely as the pressure, p.

3. A ship moving at a rate of 15 knots requires 10 hr to travel a certain distance. If the speed is increased to 25 knots, how long will the ship require to travel the same distance?

Answers:

1. $\frac{r}{k} = \frac{1}{t}$
2. $\frac{V}{k} = \frac{1}{p}$
3. 6 hr

JOINT VARIATION

A quantity **VARIES JOINTLY** as two or more quantities, if it equals a constant times their product. For example, if x, y, and z are variables and k is a constant, x varies jointly as y and z, if $x = kyz$. Note that this is similar to direct variation, except that there are two variable factors and the constant with which to contend in the one number; whereas in direct variation, we had only one variable and the constant. The equality, $x = kyz$, is equivalent to

$$\frac{x}{yz} = k$$

If a quantity varies jointly as two or more other quantities, the ratio of the first quantity to the product of the other quantities is a constant.

The formula for the area of a rectangle is an example of joint variation. If A is allowed to vary, rather than being constant as in the example used earlier in this chapter, then A varies jointly as L and W. When the formula is written for general use, it is not commonly expressed as $A = kLW$, although this is a mathematically correct form. Since the constant of proportionality in this case is 1, there is no practical need for expressing it.

Using the formula $A = LW$, we make the following observations: If $L = 5$ and $W = 3$, then $A = 3(5) = 15$. If $L = 5$ and $W = 4$, then $A = 4(5) = 20$, and so on. Changes in the area of a rectangle depend on changes in either the length or the width or both. The area varies jointly as the length and the width.

As a general example of joint variation, consider the expression $a \propto bc$. Written as an equation, this becomes $a = kbc$. If the value of a is known for particular values of b and c, we can find the new value of a corresponding to changes in the values of b and c. For example, suppose that a is 12 when b is 3 and c is 2. What is the value of a when b is 4 and c is 5? Rewriting the proportion,

$$\frac{a}{bc} = k$$

Thus

$$\frac{12}{(3)(2)} = k$$

Also,

$$\frac{a}{(4)(5)} = k$$

Since quantities equal to the same quantity are equal to each other, we can set up the following proportion:

$$\frac{a}{(4)(5)} = \frac{12}{(3)(2)}$$

$$a = 40$$

Practice problems. Using k as the constant of proportionality, write equations that express the following statements:

1. Z varies jointly as x and y.

2. S varies jointly as b times the square of r.

3. The length, W, of a radio wave varies jointly as the square root of the inductance, L, and the capacitance, C.

Answers:

1. $Z = kxy$

2. $S = kbr^2$

3. $W = k\sqrt{LC}$

COMBINED VARIATION

The different types of variation can be combined. This is frequently the case in applied problems. The equation

$$E = \frac{kw^2L}{p^2}$$

is an example of combined variation and is read, "E varies jointly as L and the square of W, and inversely as the square of p." Likewise,

$$V = \frac{krs}{t}$$

is read, "V varies jointly as r and s and inversely as t."

CHAPTER 14

DEPENDENCE, FUNCTIONS, AND FORMULAS

In chapter 13 of this course, use is made of several formulas, such as $A = LW$, $E = IR$, etc. It is the purpose of this chapter to explain the function and dependency relationships which make formulas so useful.

DEPENDENCE AND FUNCTIONS

Dependence may be defined as any relationship between two variables which allows the prediction of change in one of them as a result of change in the other. For example, the cost of 200 bolts depends upon the price per hundred. If C represents cost and p represents the price of 100 bolts, then the cost of 200 bolts may be expressed as follows:

$$C = 2p$$

In the example just given, C is called the DEPENDENT VARIABLE because its value depends upon the changing values of p. The INDEPENDENT VARIABLE is p. It is standard practice to isolate the dependent variable on the left side of an equation, as in the example.

Consider the formula for the area of a rectangle, $A = LW$. Here we have two independent variables, L and W.

Figure 14-1 (A) shows what happens if we double the length. Figure 14-1 (B) shows the result of doubling the width. Figure 14-1 (C) shows the effect of doubling both length and width. Notice that when the length or width alone is doubled the area is doubled, but when both length and width are doubled the area is four times as great.

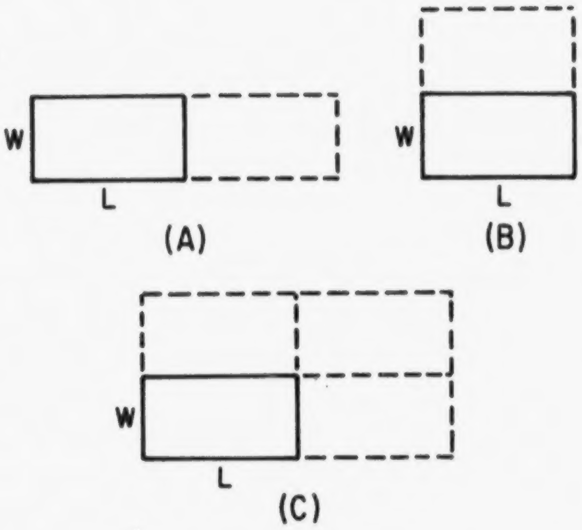


Figure 14-1.—Changes in the area of a rectangle resulting from changes in length and width.

In any equation showing a dependency relationship, the dependent variable is said to be a FUNCTION of the independent variable. Another use of the term "function" in describing an equation such as $C = 2p$ is to refer to the whole expression as "the function $C = 2p$." This terminology is especially useful when the right-hand expression has several terms. For example, consider the equation $y = 2x^2 + 3x - 4$. Mathematicians frequently use a shorthand notation and rewrite the equation as $y = f(x)$. The expression $f(x)$ is understood to mean "a function of x" and reference to the function by calling it $f(x)$ saves the space and time that would otherwise be required to write out all three terms.

Practice problems. Answer the following questions concerning the function $r = \frac{d}{t}$.

1. When t increases and d remains the same, does r increase, decrease, or remain the same?

2. When d increases and t remains the same, does r increase, decrease, or remain the same?

3. When t decreases and d remains the same, does r increase, decrease, or remain the same?

4. When d decreases and t remains the same, does r increase, decrease, or remain the same?

5. When d is doubled and t remains the same, is r doubled or halved?

6. When t is doubled and d remains the same, is r doubled or halved?

Answers:

1. Decreases.
2. Increases.
3. Increases.
4. Decreases.
5. Doubled.
6. Halved.

FORMULAS

One of the most common uses of algebra is in the solution of formulas. Formulas have a wide and varied use throughout the Navy. It is important to know how formulas are derived, how to translate them into words, how to make them from word statements, and how to use them to solve problems.

A formula is a general fact, rule, or principle expressed in algebraic symbols. It is a shorthand expression of a rule in which letters and signs of operation take the place of words. The formula always indicates the mathematical operations involved. For example, the formula $P = 2L + 2W$ indicates that the perimeter (sum of the lengths of the sides) of a rectangle is equal to twice its length plus twice its width. (See fig. 14-2.)

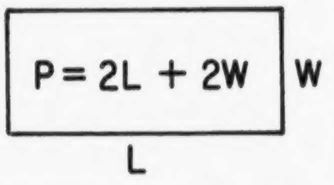


Figure 14-2.—Perimeter of a rectangle.

A formula obtained by logical or mathematical reasoning is called a mathematical formula. A formula whose reliability is based on a limited number of observations, or on immediate experience, and not necessarily on established theories or laws is called an EMPIRICAL formula. Empirical formulas are found frequently in engineering and physical sciences. They sometimes are valid for only a limited number of values.

SUBJECT OF A FORMULA

Usually a formula is taken almost directly from the verbal rule or law. For instance, the perimeter of a rectangle is equal to twice the length plus twice the width. Where possible, letters are used as symbols for the words. Thus, $P = 2L + 2W$. A simple formula such as this is like a declarative sentence. The left half is the SUBJECT and all the rest is the predicate. The subject is P. It corresponds to the part of the verbal rule that reads "the perimeter of a rectangle." This subject is usually a single letter followed by the equality sign.

All formulas are equations, but not all equations are formulas. Some distinctions between a formula and an ordinary equation are worthy of note. The equation may not have a subject, while the formula typically does. In the formula, the unknown quantity stands alone in the left-hand member. No computation is performed upon it, and it does not appear more than once. In the equation, on the other hand, the unknown quantity may appear once or more in either or both members, and computation may be performed with it or on it. We evaluate a formula by substituting for the literal numbers in the right member. An equation is solved by computation in either or both members until all that remains is an unknown in one member and a known quantity in the other. The solution of an equation usually requires a knowledge of algebraic principles, while the evaluation of a formula may ordinarily be accomplished with only a knowledge of arithmetic.

SYMBOLS

Letters that represent words have been standardized in many cases so that certain formulas may be written the same in various texts and reference books. However, to avoid any misunderstanding a short explanation often accompanies formulas as follows:

$$A = hw,$$

where

A = area in square units

h = height

w = width

Subscripts and Primes

In a formula in which two or more of the same kind of letters are being compared, it is desirable to make a distinction between them. In electronics, for example, a distinction between resistances may be indicated by R_a and R_b or R_1 and R_2. These small numbers or letters written to the right and below the R's are called subscripts. Those shown here are read: R sub a, R sub b, R sub one, and R sub two. Primes are also used in the same manner to distinguish between quantities of the same kind. Primes are written to the right and above the letters, as in S', S'', and S'''. They are read: S prime, S double prime, and S triple prime.

CHANGING THE SUBJECT OF A FORMULA

If values are given for all but one of its variables, a formula can be solved to obtain the value of that variable. The first step is usually the rearranging of the formula so that the unknown value is the subject—that is, a new formula is derived from the original. For example, the formula for linear motion—distance equals rate times time—is usually written

$$d = rt$$

Suppose that instead of the distance we wish to know the rate, r, or the time, t. We simply change the subject of the formula by the algebraic means developed in earlier chapters. Thus, in solving the formula for r, we divide both sides by t, with the following result:

$$\frac{d}{t} = \frac{rt}{t}$$

$$\frac{d}{t} = r, \text{ or } r = \frac{d}{t}$$

In words, this formula states that rate equals distance divided by time. Likewise, in solving for t, we have the following :

$$\frac{d}{r} = \frac{rt}{t}$$

$$\frac{d}{r} = t, \text{ or } t = \frac{d}{r}$$

In words, this formula states that time equals distance divided by rate.

We have in effect two new formulas, the subject of one being rate and the subject of the other being time. They are related to the original formula because they were derived from it, but they are different in that they have different subjects.

Practice problems. Derive new formulas from the following expressions with subjects as indicated:

1. $A = \frac{1}{2}bh$, subject h
2. $P = 2L + 2W$, subject L
3. $i = prt$, subject r
4. $p = br$, subject b
5. $E = IR$, subject I
6. The modern formula for converting Fahrenheit temperatures to Celsius (centigrade) is $C = (F + 40)\left(\frac{5}{9}\right) - 40$. Express the formula for converting Celsius (centigrade) temperatures to Fahrenheit.

Answers:

1. $h = \frac{2A}{b}$
2. $L = \frac{P - 2W}{2}$
3. $r = \frac{i}{pt}$
4. $b = \frac{p}{r}$
5. $I = \frac{E}{R}$
6. $F = (C + 40)\ \frac{9}{5} - 40$

EVALUATING FORMULAS

The first step in finding the value of the unknown variable of a formula is usually the derivation of a formula that has the unknown as its subject. Once this is accomplished, the evaluation of a formula consists of nothing more than substituting numerical values for the letters representing known quantities and performing the indicated operations.

For example, suppose we wish to find the time required to fly 1,250 nautical miles at the rate of 250 knots. The formula is d = rt. We can change the subject by dividing both sides of the equation by r, as follows:

$$\frac{d}{r} = \frac{rt}{r}$$

$$\frac{d}{r} = t$$

$$t = \frac{1250}{250} = 5 \text{ hr}$$

Formulas can be solved for an unknown by substituting directly in the original formula even though that unknown is not the subject. Generally, however, it is simpler to first make the unknown the subject.

Formulas vary widely, from the simple type such as we have been considering to some that are very complex. All formulas have certain characteristics in common. There is always a subject, the quantity whose value is sought as a final answer. This subject usually stands alone, being placed equal to at least one and possibly several literal numbers, which are combined according to certain indicated operations. The formula can always be evaluated for a specific case when numerical values are known for all these literal quantities.

Evaluating formulas may be facilitated by developing a routine order of doing the work. If someone else can read the work and clearly understand what has been done, the work is in good order. The original formula should be written first, then the derived formula that is going to be used in solving the problem, and finally the actual substitutions. The indicated operations may then be carried out. Care should be taken to label answers with correct units; that is, miles per hour, foot-pounds, square feet, etc.

Practice problems.

1. $E = IR$. Solve for R in ohms if E is 110 volts and I is 5 amperes.

2. $d = rt$. Solve for t in hours if d is 840 nautical miles and r is 25 knots.

3. $F = (C + 40)\left(\frac{9}{5}\right) - 40$. Solve for C if F is 32°.

Answers:

1. 22 ohms
2. 33.6 hr
3. 0°

DEVELOPING FORMULAS

Developing a formula from a verbal statement is nothing more than reducing the statement to a shorthand form and showing the mathematical relationships between the elements of the statement.

For example, suppose that we wish to develop a formula showing the distance, D, traveled at the rate of 20 knots for t hours. If the distance traveled in 1 hr is 20 nautical miles, then the distance traveled in t hours is 20t. Therefore, the formula is

$$D = 20t$$

Practice problems.

1. Write a formula for the cost, C of p pounds of sugar at 15 cents per pound.

2. Write the formula for the cost, C, of one article when the total cost, T, of n similar articles is known.

3. Write a formula for the number of days, d, in w weeks.

4. Write a formula for the number of ounces, n, in p pounds.

Answers:

1. $C = 15p$
2. $C = T/n$
3. $d = 7w$
4. $n = 16p$

Developing Formulas from Tables

In technical work, instrument readings and other data are often recorded in a tabular arrangement. By careful observation of such tables of data, it is frequently possible to find values that are related in a definite pattern. The table can thus be used in developing a formula showing the relationship between the related quantities.

For example, table 14-1 shows the results of time trials on a ship, with the data rounded to the nearest whole hour and the nearest whole mile.

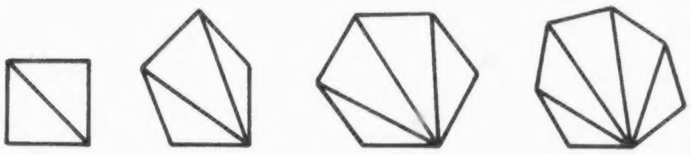
Table 14-1.—Time trials.

Nautical miles (d)	20	40	60	80	100
Hours (t)	1	2	3	4	5

By inspection of the table, it soon becomes clear that the number of miles traveled is always 20 times the corresponding number of hours. Therefore the formula developed from this table is as follows:

$$d = 20t$$

A second example of the derivation of a formula from a table is shown in figure 14-3. Figure 14-3 (A) shows several polygons (many-sided plane figures), each with one or more diagonals. A diagonal is a straight line joining one vertex (point where two sides meet) with another.



(A)

DIAGONALS (d)	1	2	3	4
SIDES (n)	4	5	6	7

(B)

Figure 14-3.—Diagonals of plane figures.

The table in figure 14-3 (B) compares the number of sides of each polygon with the number of diagonals that can be drawn from any one vertex. Using this table, we make a formula for the number, d, of diagonals that can be drawn from one vertex of a polygon of n sides. In the table we note that the number of diagonals is always 3 less than the number of sides. Therefore the formula is $d = n - 3$.

Practice problems. Complete the following tables and write formulas to show the relationship between the numbers.

1.

L	2	5	8	11	14	17	20
P	12	30	48	66	84		

2.

a	0	1	2	3	4	5	6	7
b	4	5	6	7				

3.

x	0	1	2	3	4	5	6
y	0	3	6	9	12		

4.

n	1	2	3	4	5	6	7
s	3	6	11	18	27	38	

Answers:

1. $P = 6L$
2. $b = a + 4$
3. $y = 3x$
4. $s = n^2 + 2$

TRANSLATING FORMULAS

Thus far, we have been concerned primarily with reducing verbal rules or statements to formula form. It is also necessary to be able to do the reverse, and translate a formula into words. Technical publications frequently take advantage of the fact that it is more convenient to write formulas than longhand rules. Understanding is hampered if we are not able to translate these formulas into words. As an example of translation, we may translate the formula $V = lwh$ into words, with the literal factors representing words as follows:

V = volume of a rectangular solid
l = length
w = width
h = height

This produces the following translation: The volume of a rectangular solid equals the length times the width times the height.

As a second example, we translate the algebraic expression $2\sqrt{x} - 4$ into words as follows: Twice the square root of a certain number, minus 4.

Practice problems. Translate each of the following expressions into words.

1. $PV = k$, where P represents pressure of a gas and V represents volume. (Assume constant temperature.)

2. $x = y + 4$, where x and y are numbers.

3. $A = LW$, where A is the area of a rectangle, L is its length, and W is its width.

4. $d = rt$, where d is distance, r is rate, and t is time.

Answers:

1. The pressure of a gas multiplied by its volume is constant, if the temperature is constant.

2. A certain number, x, is equal to the sum of another number, y, and 4.

3. The area of a rectangle is equal to the product of its length times its width.

4. Distance is equal to rate multiplied by time.

GRAPHING FORMULAS

We have seen that the formula is an equation. Since all formulas are equations they may be graphed. Graphs of formulas have wide use in the Navy in such fields as electronics and engineering. In practical applications it is often convenient to derive information from graphs of formulas rather than from formulas directly.

As an example, suppose that a fuel costs 30 cents per gallon. The formula for the cost in dollars of n gallons is

$$C = 0.30n$$

We see that this is a linear equation, the resulting curve of which passes through the origin (no constant term). Since we are interested only in positive values, we can eliminate three quadrants of the graph and use only the first quadrant. We already know one point on the graph is (0,0). We need plot only one other point to graph the formula. The result is shown in figure 14-4.

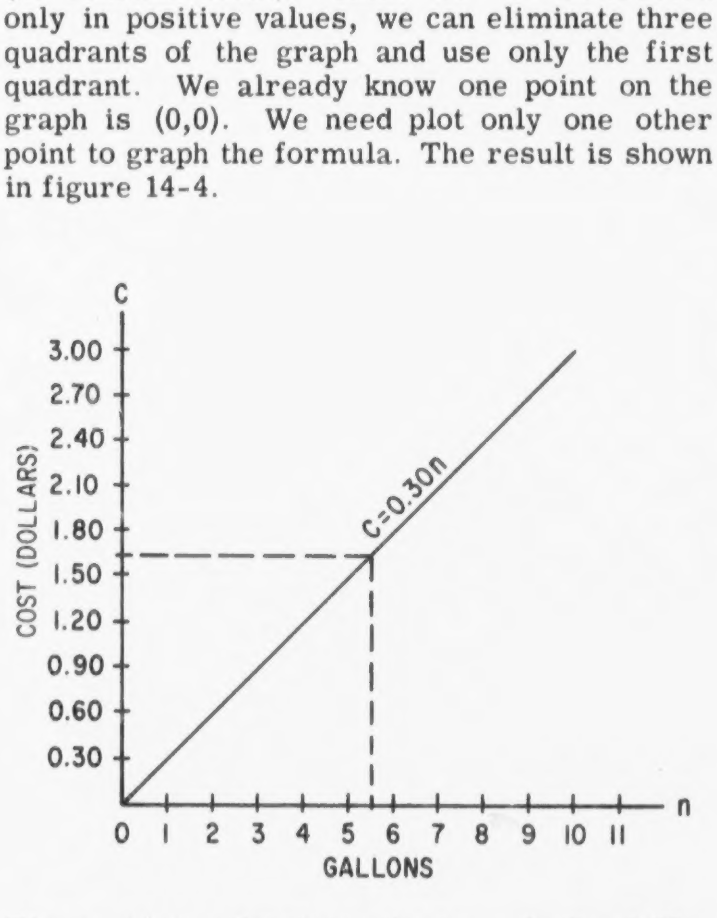


Figure 14-4.—Graph for the formula C = 0.30n.

We may read the cost directly from the graph when the number of gallons is known, or the number of gallons when the cost is known. For instance, if 5-1/2 gal are sold, find 5-1/2 on the gallons scale and follow the vertical line from that point to the point where it intersects the graph of the formula. From this point, follow the horizontal line to the cost scale. The horizontal line intersects the cost scale at 1.65. Therefore the cost of 5-1/2 gal is $1.65.

Likewise, to answer the question, "How many gallons may be bought for $1.27," we would enlarge the graph enough to estimate to the exact cent. Then we would follow a horizontal line from 1.27 on the cost scale to the formula graph and follow a vertical line from that point to the gallons scale. Thus, 4-1/4 gal may be bought for $1.27.

Plotting two formulas on the same graph may help to solve certain kinds of problems. For example, suppose that two ships leave port at the same time. One averages 10 knots and the other averages 15 knots. How far has each traveled at the end of 3 hr and at the end of 5 hr? A graph to relate the two ships' movements at any time can be made as follows: Let the vertical scale be in nautical miles and the

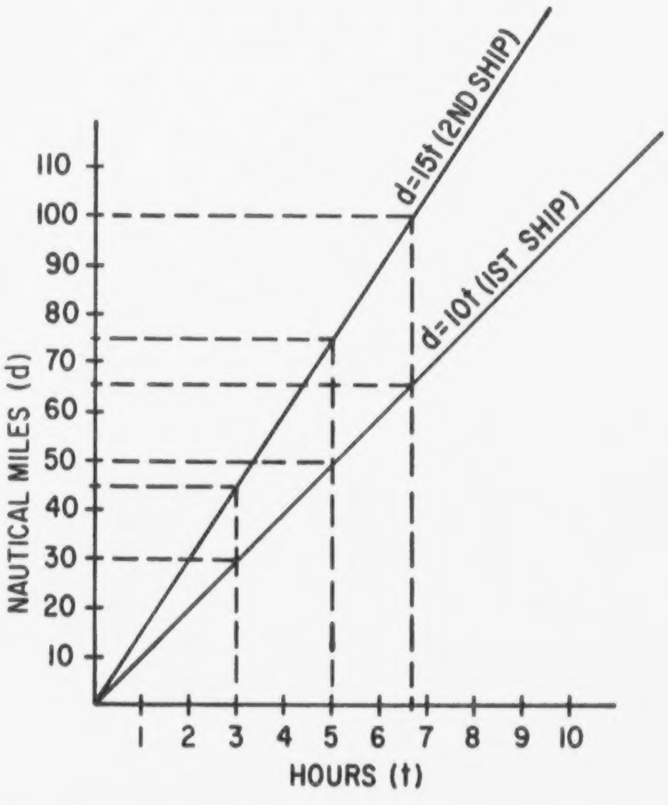


Figure 14-5.—Graph of the formulas d = 10t and d = 15t.

horizontal scale be in hours. The formula for the first ship's distance related to time is

$$d = 10t$$

The formula for the second ship's distance related to time is

$$d = 15t$$

We see that these formulas are linear and their curves pass through the origin. They are graphed in figure 14-5.

With this graph we can now answer the questions originally posed, at a glance. Thus in 3 hr the first ship traveled 30 mi and the second traveled 45 mi. In 5 hr the first ship traveled 50 mi and the second traveled 75 mi.

We could also answer such questions as: When the second ship has traveled 100 mi, how far has the other traveled? We first find the point on the graph of $d = 15t$ where the ship has traveled 100 mi. We then follow the vertical line from that point to the point where it intersects the graph of the other formula. From the point of intersection we follow a horizontal line to the distance axis and see that the first ship has traveled about 67 mi when the second has traveled 100 mi.

The foregoing examples serves to illustrate the wide variety of applications in which graphs of formulas are useful.

CHAPTER 15

COMPLEX NUMBERS

In certain calculations in mathematics and related sciences, it is necessary to perform operations with numbers unlike any mentioned thus far in this course. These numbers, unfortunately called "imaginary" numbers by early mathematicians, are quite useful and have a very real meaning in the physical sense. The number system, which consists of ordinary numbers and imaginary numbers, is called the COMPLEX NUMBER system. Complex numbers are composed of a "real" part and an "imaginary" part.

This chapter is designed to explain imaginary numbers and to show how they can be combined with the numbers we already know.

REAL NUMBERS

The concept of number, as has been noted in previous chapters, has developed gradually. At one time the idea of number was limited to positive whole numbers.

The concept was broadened to include positive fractions; numbers that lie between the whole numbers. At first, fractions included only those numbers which could be expressed with terms that were integers. Since any fraction may be considered as a ratio, this gave rise to the term RATIONAL NUMBER, which is defined as any number which can be expressed as the ratio of two integers. (Remember that any whole number is an integer.)

It soon became apparent that these numbers were not enough to complete the positive number range. The ratio, π, of the circumference of a circle to its diameter, did not fit the concept of number thus far advanced, nor did such numbers as $\sqrt{2}$ and $\sqrt{3}$. Although decimal values are often assigned to these numbers, they are only approximations. That is, π is not exactly equal to 22/7 or to 3.142. Such numbers are called IRRATIONAL to distinguish them from the other numbers of the system. With rational and irrational numbers, the positive number system includes all the numbers from zero to infinity in a positive direction.

Since the number system was not complete with only positive numbers, the system was expanded to include negative numbers. The idea of negative rational and irrational numbers to minus infinity was an easy extension of the system.

Rational and irrational numbers, positive and negative to ± infinity as they have been presented in this course, comprise the REAL NUMBER system. The real number system is pictured in figure 15-1.

OPERATORS

As shown in a previous chapter, the plus sign in an expression such as 5 + 3 can stand for either of two separate things: It indicates the positive number 3, or it indicates that +3 is to be added to 5; that is, it indicates the operation to be performed on +3.

Likewise, in the problem 5 - 3, the minus sign may indicate the negative number -3, in which case the operation would be addition; that is, 5 + (-3). On the other hand, it may indicate the sign of operation, in which case +3 is to be subtracted from 5; that is, 5 - (+3).

Thus, plus and minus signs may indicate positive and negative numbers, or they may indicate operations to be performed.

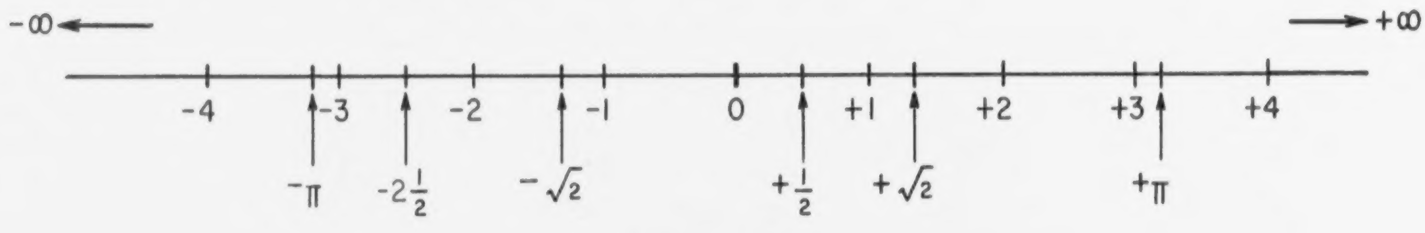


Figure 15-1.—The real number system.

IMAGINARY NUMBERS

The number line pictured in figure 15-1 represents all positive and negative numbers from plus infinity to minus infinity. However, there is a type of number which does not fit into the picture. Such a number occurs when we try to solve the following equation:

$$x^2 + 4 = 0$$
$$x^2 = -4$$
$$x = \pm \sqrt{-4}$$

Notice the distinction between this use of the radical sign and the manner in which it was used in chapter 7. Here, the ± symbol is included with the radical sign to emphasize the fact that two values of x exist. Although both roots exist, only the positive one is usually given. This is in accordance with usual mathematical convention.

The equation

$$x = \pm \sqrt{-4}$$

raises an interesting question:

What number multiplied by itself yields -4? The square of -2 is +4. Likewise, the square of +2 is +4. There is no number in the system of real numbers that is the square root of a negative number. The square root of a negative number came to be called an IMAGINARY NUMBER. When this name was assigned the square roots of negative numbers, it was natural to refer to the other known numbers as the REAL numbers.

IMAGINARY UNIT

To reduce the problem of imaginary numbers to its simplest terms, we proceed as far as possible using ordinary numbers in the solution. Thus, we may write $\sqrt{-4}$ as a product

$$\begin{aligned} \sqrt{-1.4} &= \sqrt{4}\sqrt{-1} \\ &= \pm 2\sqrt{-1} \end{aligned}$$

Likewise,

$$\sqrt{-5} = \sqrt{5}\sqrt{-1}$$

Also,

$$3\sqrt{-7} = 3\sqrt{-7}\sqrt{-1}$$

Thus, the problem of giving meaning to the square root of any negative number reduces to that of finding a meaning for $\sqrt{-1}$.

The square root of minus 1 is designated i by mathematicians. When it appears with a coefficient, the symbol i is written last unless the coefficient is in radical form. This convention is illustrated in the following examples:

$$\pm 2\sqrt{-1} = \pm 2i$$
$$\sqrt{5}\sqrt{-1} = i\sqrt{5}$$
$$3\sqrt{7}\sqrt{-1} = 3i\sqrt{7}$$

The symbol i stands for the imaginary unit $\sqrt{-1}$. An imaginary number is any real multiple, positive or negative, of i. For example, $-7i$, $+7i$, $i\sqrt{15}$, and bi are all imaginary numbers.

In electrical formulas the letter i denotes current. To avoid confusion, electronic technicians use the letter j to indicate $\sqrt{-1}$ and call it "operator j." The name "imaginary" should be thought of as a technical mathematical term of convenience. Such numbers have a very real purpose in the physical sense. Also it can be shown that ordinary mathematical operations such as addition, multiplication, and so forth, may be performed in exactly the same way as for the so-called real numbers.

Practice problems. Express each of the following as some real number times i:

1. $\sqrt{-16}$
2. $2\sqrt{-1}$
3. $\sqrt{-5}$
4. $\frac{d}{f}\sqrt{-f^2}$
5. $\sqrt{-25}$
6. $\sqrt{-\frac{9}{16}}$

Answers:

1. $4i$
2. $2i$
3. $i\sqrt{5}$
4. di
5. $5i$
6. $\frac{3}{4}i$

Powers of the Imaginary Unit

The following examples illustrate the results of raising the imaginary unit to various powers:

$$i = \sqrt{-1}$$
$$i^2 = \sqrt{-1}\sqrt{-1}, \text{ or } -1$$
$$i^3 = i^2 i = -1i, \text{ or } -i$$
$$i^4 = i^2 i^2 = -1 \cdot -1 = +1$$
$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

We see from these examples that an even power of i is a real number equal to +1 or -1. Every odd power of i is imaginary and equal to i or -i. Thus, all powers of i reduce to one of the following four quantities: $\sqrt{-1}$, -1, $-\sqrt{-1}$, or +1.

GRAPHICAL REPRESENTATION

Figure 15-1 shows the real numbers represented along a straight line, the positive numbers extending from zero to the right for an infinite distance, and the negative numbers extending to the left of zero for an infinite distance. Every point on this line corresponds to a real number, and there are no gaps between them. It follows that there is no possibility of representing imaginary numbers on this line.

Earlier, we noted that certain signs could be used as operators. The plus sign could stand for the operation of addition. The minus sign could stand for the operation of subtraction. Likewise, it is easy to explain the imaginary number i graphically as an operator indicating a certain operation is to be performed on the number of which it is the coefficient.

If we graphically represent the length, n, on the number line pictured in figure 15-2 (A), we start at the point 0 and measure to the right (positive direction) a distance representing n units. If we multiply n by -1, we may represent the result -n by measuring from 0 in a negative direction a distance equal to n units.

Graphically, multiplying a real number by -1 is equivalent to rotating the line that represents the number about the point 0 through 180° so that the new position of n is in the opposite direction and a distance n units from 0. In this case we may think of -1 as the operator that rotates n through two right angles to its new position (fig. 15-2 (B)).

As we have shown, $i^2 = -1$. Therefore, we have really multiplied n by i^2, or i x i. In other words, multiplying by -1 is the same as multiplying by i twice in succession. Logically, if we multiplied n by i just once, the line n would be rotated only half as much as before—that is, through only one right angle, or 90°. The new segment ni would be measured in a direction 90° from the line n. Thus, i is an operator that rotates a number through one right angle. (See fig. 15-3.)

We have shown previously that a positive number may have two real square roots, one positive and one negative. For example, $\sqrt{9} = \pm 3$.

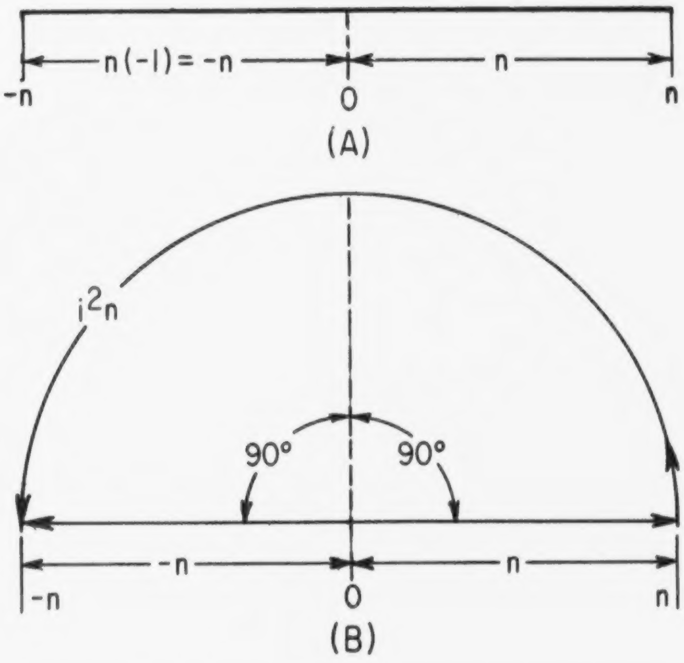


Figure 15-2.—Graphical multiplication by -1 and by operator i^2.

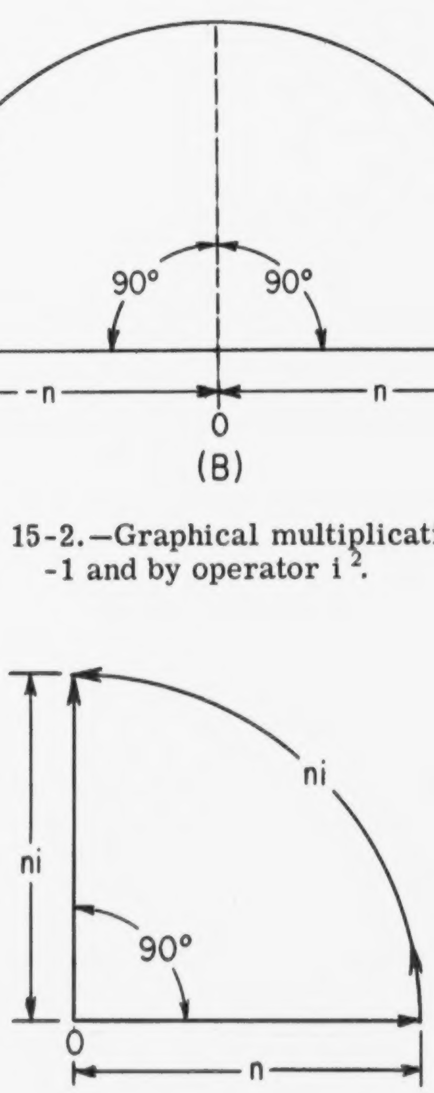


Figure 15-3.—Graphical multiplication by operator i.

We also saw that an imaginary number may have two roots. For example, $\sqrt{-4}$ is equal to ±2i. When the operator -1 graphically rotates a number, it may do so in a counterclockwise or a clockwise direction. Likewise, the operator i may graphically rotate a number in either direction. This fact gives meaning to numbers such as ±2i. It has been agreed that a number multiplied by +i is to be rotated 90° in a counterclockwise direction. A number multiplied by -i is to be rotated 90° in a clockwise direction.

In figure 15-4, +2i is represented by rotating the line that represents the positive real number 2 through 90° in a counterclockwise direction. It follows that -2i is represented by rotating the line that represents the positive real number 2 through 90° in a clockwise direction.

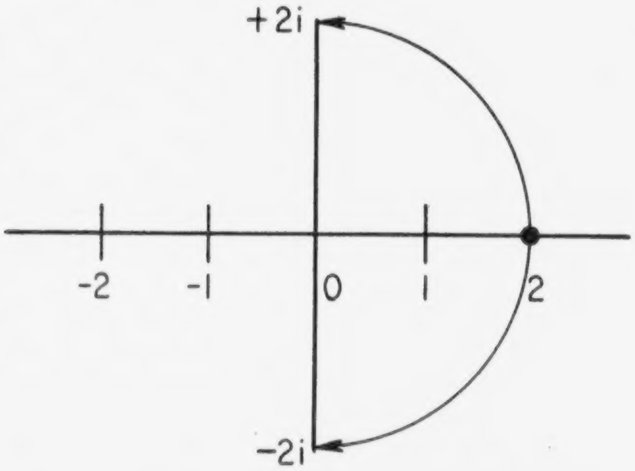


Figure 15-4.—Graphical representation of ±2i.

In figure 15-5, notice that the idea of i as an operator agrees with the concept advanced concerning the powers of i. Thus, i rotates a number through 90°; i^2 or -1 rotates the number through 180°, and the number is real and negative; i^3 rotates the number through 270°, which has the same effect as -1; and i^4 rotates the number through 360°, and the number is once again positive and real.

THE COMPLEX PLANE

All imaginary numbers may be represented graphically along a line extending through zero and perpendicular to the line representing the real numbers. This line may be considered infinite in both the positive and negative directions, and all multiples of i may be represented on it. This graph is similar to the rectangular coordinate system studied earlier.

In this system, the vertical or y axis is called the axis of imaginaries, and the horizontal or x axis is called the axis of reals. In the rectangular coordinate system, real numbers are laid off on both the x and y axes and the plane on which the axes lie is called the real plane. When the y axis is the axis of imaginaries, the plane determined by the x and y axes is called the COMPLEX PLANE (fig. 15-6).

In any system of numbers a unit is necessary for counting. Along the real axis, the unit is the number 1. As shown in figure 15-6, along the imaginary axis the unit is i. Numbers that lie along the imaginary axis are called PURE IMAGINARIES. They will always be some multiple of i, the imaginary unit. The numbers 5i, 3i $\sqrt{2}$, and $\sqrt{-7}$ are examples of pure imaginaries.

NUMBERS IN THE COMPLEX PLANE

All numbers in the complex plane are complex numbers, including reals and pure imaginaries. However, since the reals and imaginaries have

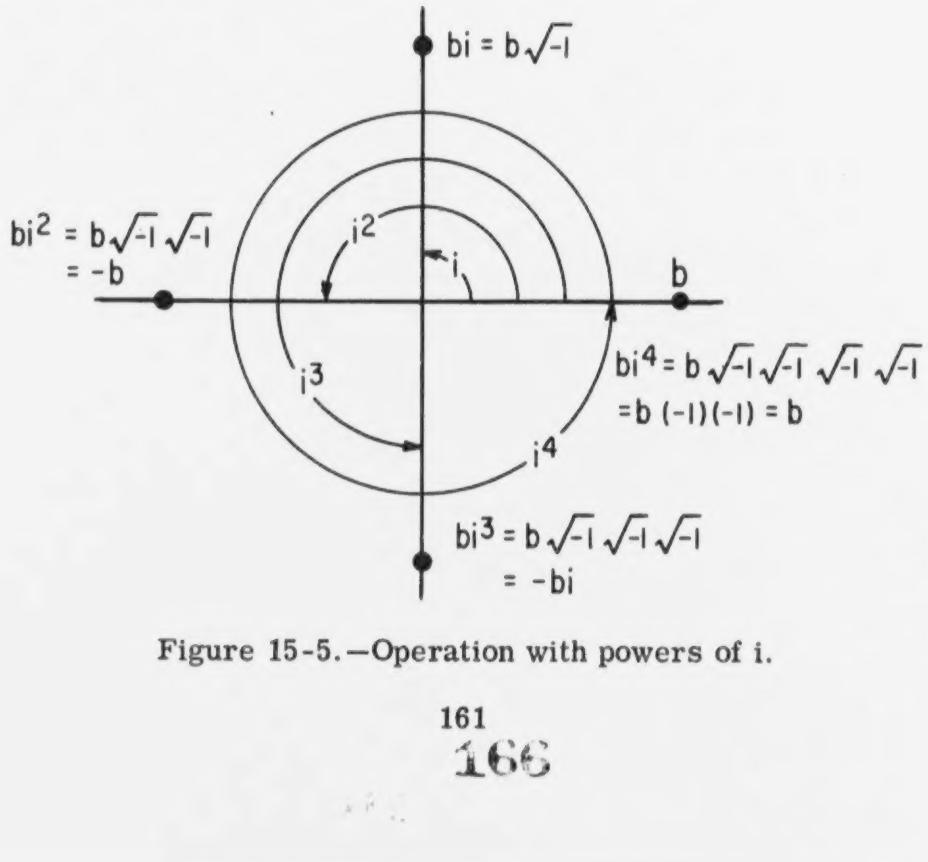

$bi = b\sqrt{-1}$
$bi^2 = b\sqrt{-1}\sqrt{-1}$
$= -b$
i^2
i
b
i^3
$bi^4 = b\sqrt{-1}\sqrt{-1}\sqrt{-1}\sqrt{-1}$
$= b(-1)(-1) = b$
i^4
$bi^3 = b\sqrt{-1}\sqrt{-1}\sqrt{-1}$
$= -bi$

Figure 15-5.—Operation with powers of i.

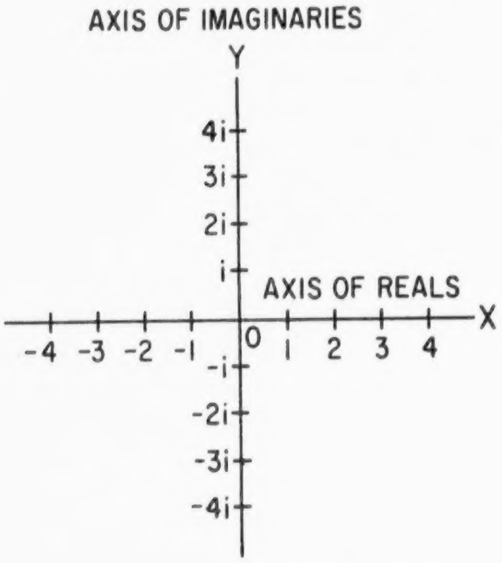

AXIS OF IMAGINARIES
Y
4i
3i
2i
i
AXIS OF REALS
X
-4
-3
-2
-1
0
1
2
3
4
-i
-2i
-3i
-4i

Figure 15-6.—The complex plane.

the special property of being located on the axes, they are usually identified by their distinguishing names.

The term complex number has been defined as the indicated sum or difference of a real number and an imaginary number.

For example, $3 + 5\sqrt{-1}$ or $3 + 5i$, $2 - 6i$, and $-2 + \sqrt{-5}$ are complex numbers. In the complex number $7 - i\sqrt{2}$, 7 is the real part and $-1\sqrt{2}$ is the imaginary part.

All complex numbers correspond to the general form $a + bi$, where a and b are real numbers. When a has the value 0, the real term disappears and the complex number becomes a pure imaginary. When b has the value of 0, the imaginary term disappears and the complex number becomes a real number. Thus, 4 may be thought of as $4 + 0i$, and 3i may be considered $0 + 3i$. From this we may reason that the real number and the pure imaginary number are special cases of the complex number. Consequently, the complex number may be thought of as the most general form of a number and can be construed to include all the numbers of algebra as shown in the chart in figure 15-7.

Plotting Complex Numbers

Complex numbers may easily be plotted in the complex plane. Pure imaginaries are plotted along the vertical axis, the axis of imaginaries, and real numbers are plotted along the horizontal axis, the axis of reals. It follows that other points in the complex plane must represent numbers that are part real and part imaginary; in other words, complex numbers. If we wish to plot the point $3 + 2i$, we note that the number is made up of the real number 3 and the imaginary number 2i. Thus, as in figure 15-8, we measure along the real axis in a

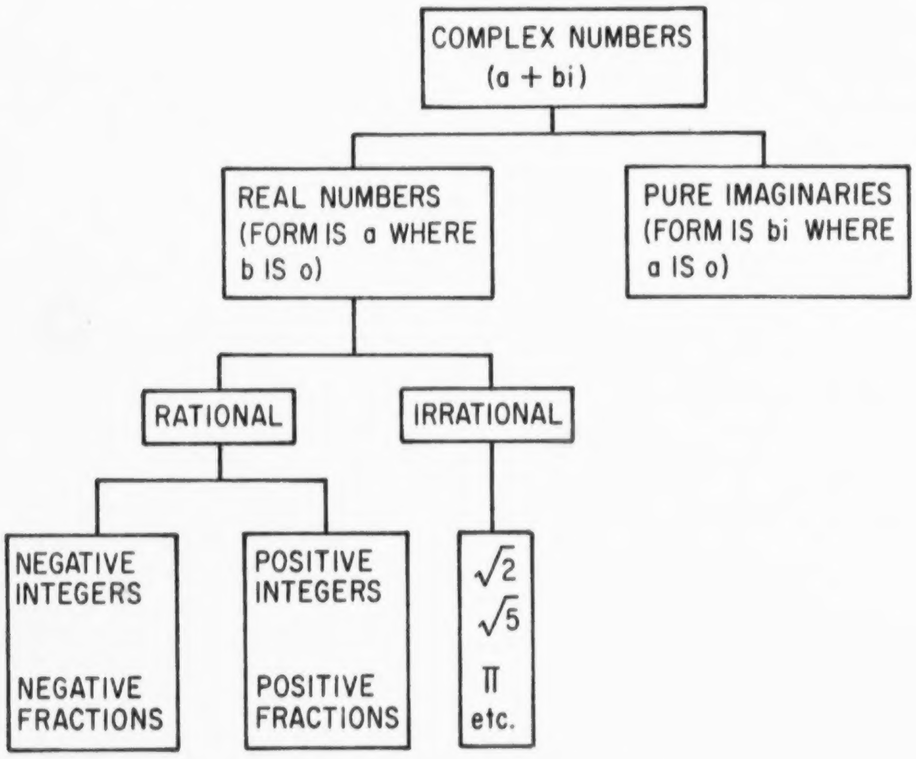

COMPLEX NUMBERS
(a + bi)
REAL NUMBERS
(FORM IS a WHERE
b IS o)
PURE IMAGINARIES
(FORM IS bi WHERE
a IS o)
RATIONAL
IRRATIONAL
NEGATIVE
INTEGERS
NEGATIVE
FRACTIONS
POSITIVE
INTEGERS
POSITIVE
FRACTIONS
√2
√5
π
etc.

Figure 15-7.—The complex number system.

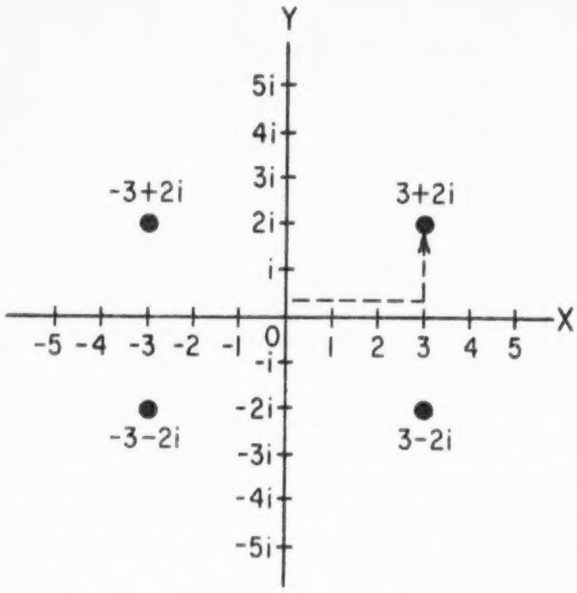


Figure 15-8.—Plotting complex numbers.

positive direction. At point (3, 0) on the real axis we turn through one right angle and measure 2 units up and parallel to the imaginary axis. Likewise, the number -3 + 2i is 3 units to the left and up 2 units; the number 3 - 2i is 3 units to the right and down 2 units; and the number -3 -2i is 3 units to the left and down 2 units.

Complex Numbers as Vectors

A vector is a directed line segment. A complex number represents a vector expressed in the RECTANGULAR FORM. For example, the complex number 6 + 8i in figure 15-9 may be considered as representing either the point P or the line OP. The real parts of the complex number (6 and 8) are the rectangular components of the vector. The real parts are the legs of the right triangle (sides adjacent to the right angle), and the vector OP is its hypotenuse (side opposite the right angle). If we merely wish to indicate the vector OP, we may do so by writing the complex number that represents it along the segment as in figure 15-9. This method not only fixes the position of point P, but also shows what part of the vector is imaginary (PA) and what part is real (OA).

If we wish to indicate a number that shows the actual length of the vector OP, it is necessary to solve the right triangle OAP for its hypotenuse. This may be accomplished by taking the square root of the sum of the squares of

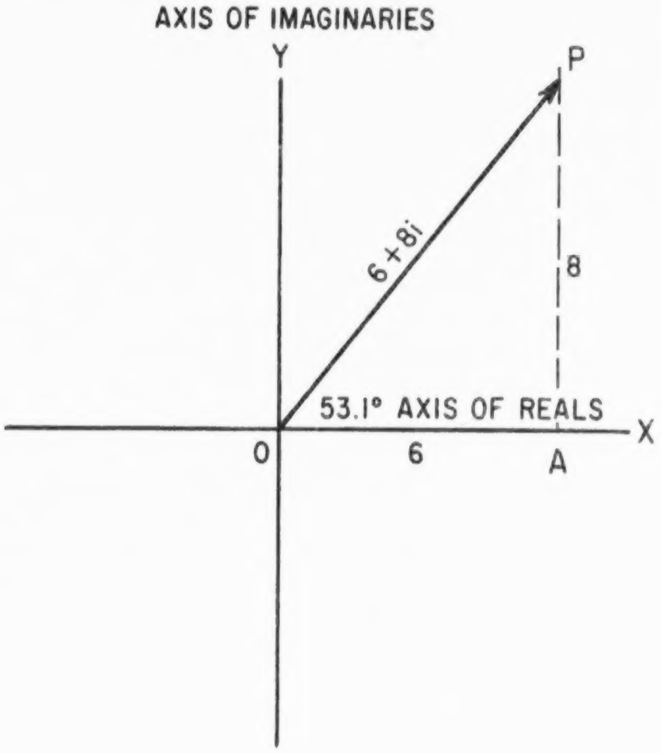


Figure 15-9.—A complex number shown as a vector.

the legs of the triangle, which in this case are the real numbers, 6 and 8. thus,

$$\begin{aligned} OP &= \sqrt{6^2 + 8^2} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

However, since a vector has direction as well as magnitude, we must also show the direction of the segment; otherwise the segment OP could radiate in any direction on the complex plane from point 0. The expression $10\underline{/53.1^\circ}$ indicates that the vector OP has been rotated counterclockwise from the initial position through an angle of 53.1°. (The initial position in a line extending from the origin to the right along OX.) This method of expressing the vector quantity is called the POLAR FORM. The number represents the magnitude of the quantity, and the angle represents the position of the vector with respect to the horizontal reference, OX. Positive angles represent counterclockwise rotation of the vector, and negative angles represent clockwise rotation. The polar form is generally simpler for multiplication and division, but its use requires a knowledge of trigonometry.

ADDITION AND SUBTRACTION OF COMPLEX NUMBERS

Pure imaginaries are added and subtracted in the same way as any other algebraic quantities. The coefficients of similar terms are added or subtracted algebraically, as follows:

$$4i + 3i = 7i$$
$$4i - 3i = i$$
$$4i - (-3i) = 7i$$

Likewise, complex numbers in the rectangular form are combined like any other algebraic polynomials. Add or subtract the coefficients of similar terms algebraically. If parentheses enclose the numbers, first remove the parentheses. Next, place the real parts together and the imaginary parts together. Collect terms. As examples, consider the following:

$$1.\ (2 - 3i) + (5 + 4i) = 2 - 3i + 5 + 4i$$
$$= 2 + 5 - 3i + 4i$$
$$= 7 + i$$

$$2.\ (2 - j3) - (5 + j4) = 2 - j3 - 5 - j4$$
$$= 2 - 5 - j3 - j4$$
$$= -3 - j7$$

In example 2, notice that the convention for writing operator j (the electronics form of the imaginary unit) with numerical coefficients is to place j first.

If the complex numbers are placed one under the other, the results of addition and subtraction appear as follows:

$$\begin{array}{cc} \text{ADDITION} & \text{SUBTRACTION} \\ 3 + 4\sqrt{-1} & a + jb \\ \underline{2 - 7\sqrt{-1}} & \underline{[\leftarrow c + jd} \\ 5 - 3\sqrt{-1} & (a - c) + j(b - d) \end{array}$$

Practice problems. Add or subtract as indicated, in the following problems:

1. $(3a + 4i) + (0 - 2i)$
2. $(3 + 2i) + (-3 + 3i)$
3. $(a + bi) + (c + di)$
4. $(1 + 2\sqrt{-1}) + (-2 - 2\sqrt{-1})$
5. $(-5 + 3i) - (4 - 2i)$
6. $(a + bi) - (-c + di)$

Answers:

1. $3a + 2i$
2. $5i$
3. $a + c + (b + d)i$
4. -1
5. $-9 + 5i$
6. $a + c + (b - d)i$

MULTIPLICATION OF COMPLEX NUMBERS

Generally, the rules for the multiplication of complex numbers and pure imaginaries are the same as for other algebraic quantities. However, there is one exception that should be noted: The rule for multiplying numbers under radical signs does not apply to TWO NEGATIVE numbers. When at least one of two radicands is positive, the radicands can be multiplied immediately, as in the following examples:

$$\sqrt{2}\sqrt{3} = \sqrt{6}$$
$$\sqrt{2}\sqrt{-3} = \sqrt{-6}$$

When both radicands are negative, however as in $\sqrt{-2}\sqrt{-3}$, an inconsistent result is obtained if we multiply both numbers under th radical signs immediately. To get the correc result, express the imaginary numbers first i terms of i, as follows:

$$\sqrt{-2}\sqrt{-3} = i\sqrt{2} \cdot i\sqrt{3}$$
$$= i^2\sqrt{2}\sqrt{3}$$
$$= i^2\sqrt{6}$$
$$= (-1)\sqrt{6} = -\sqrt{6}$$

Multiplying complex numbers is equivalen to multiplying binomials in the manner explained previously. After the multiplication i performed, simplify the powers of i as in th following examples:

$$1.\ \begin{array}{l} 4 - i \\ \underline{3 + i} \\ 12 - 3i \\ \underline{\quad + 4i - i^2} \\ 12 + i - i^2 = 12 + i - (-1) \\ \qquad\qquad\quad = 13 + i \end{array}$$

2. $(-6 + 5\sqrt{-7})\ (8 - 2\sqrt{-7})$

$$= (-6 + 5i\sqrt{7})\ (8 - 2i\sqrt{7})$$
$$= -48 + 40i\sqrt{7} + 12i\sqrt{7} - 10(7)i^2$$
$$= -48 + 52i\sqrt{7} + 70$$
$$= 22 + 52i\sqrt{7}$$

Practice problems. Perform the indicated operations:

1. $\sqrt{-9}\ \sqrt{-16}$
2. $\sqrt{-2}\ \sqrt{18}$
3. $\sqrt{-9}\ \sqrt{-4}$
4. $a\sqrt{-ba} \cdot \sqrt{-b}$
5. $(2 + 5i)\ (3 - 2i)$
6. $(a + \sqrt{-b})\ (a - \sqrt{-b})$
7. $(-2 + \sqrt{-4})\ (-1 + \sqrt{-4})$
8. $(8 - \sqrt{-7})\ (6 + \sqrt{-7})$

Answers:

1. -12
2. $6i$
3. 6
4. $-ab\sqrt{a}$
5. $16 + 11i$
6. $a^2 + b$
7. $-2 - 6i$
8. $55 + 2i\sqrt{7}$

CONJUGATES AND SPECIAL PRODUCTS

Two complex numbers that are alike except for the sign of their imaginary parts are called CONJUGATE COMPLEX NUMBERS. For example, $3 + 5i$ and $3 - 5i$ are conjugates. Either number is the conjugate of the other.

If one complex number is known, the conjugate can be obtained immediately by changing the sign of the imaginary part. The conjugate of $-8 + \sqrt{-10}$ is $-8 - \sqrt{-10}$. The conjugate of $-\sqrt{-6}$ is $\sqrt{-6}$.

The sum of two conjugate complex numbers is a real number, as illustrated by the following:

1. $(3 + j5) + (3 - j5) = 2(3) = 6$

2. $\left(-\frac{1}{2} + \frac{\sqrt{-3}}{2}\right) + \left(-\frac{1}{2} - \frac{\sqrt{-3}}{2}\right)$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} - \frac{\sqrt{3}}{2}i$$
$$= -\frac{1}{2} + \left(-\frac{1}{2}\right)$$
$$= -1$$

Product of Two Conjugates

The product of two conjugate complex numbers is a real number. Multiplying two conjugates is equivalent to finding the product of the sum and difference of two numbers.

Consider the following examples:

1. $(3 + j5)\ (3 - j5) = 3^2 - (j5)^2$

$$= 9 - 25(-1)$$
$$= 9 + 25$$
$$= 34$$

2. $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = \left(-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^2$

$$= \frac{1}{4} - \left[\frac{3}{4}(-1)\right]$$
$$= \frac{1}{4} + \frac{3}{4}$$
$$= 1$$

Squaring a Complex Number

Squaring a complex number is equivalent to raising a binomial to the second power. For example:

$$(-6 - \sqrt{-25})^2 = (-6 - j5)^2$$
$$= [(-1) \cdot (6 + j5)]^2$$
$$= (-1)^2 \cdot (6^2 + j60 + j^2 25)$$
$$= 36 + j60 - 25$$
$$= 11 + j60$$

DIVISION OF COMPLEX NUMBERS

When dividing by a pure imaginary, the denominator may be rationalized and the problem thus simplified by multiplying both numerator and denominator by the denominator. Thus,

$$\frac{12}{\sqrt{-2}} = \frac{12}{i\sqrt{2}} \cdot \frac{i\sqrt{2}}{i\sqrt{2}}$$
$$= \frac{12i\sqrt{2}}{2i^2}$$
$$= \frac{6i\sqrt{2}}{-1}$$
$$= -6i\sqrt{2}$$

Division of complex numbers can be accomplished by multiplying the numerator and denominator by the number that is the conjugate of the denominator. This process is similar to the process of rationalizing a denominator in the case of real numbers that are irrational.

As an example, consider

$$\frac{5 - 2i}{3 + i}$$

The denominator is 3 + i. Its conjugate is 3 - i. Multiplying numerator and denominator by 3 - i gives

$$\begin{aligned} \frac{5 - 2i}{3 + i} \cdot \frac{3 - i}{3 - i} &= \frac{15 - 11i + 2i^2}{9 - i^2} \\ &= \frac{15 - 11i - 2}{9 + 1} \\ &= \frac{13 - 11i}{10} \\ &= \frac{13}{10} - \frac{11}{10}i \end{aligned}$$

Practice problems. Rationalize the denominators and simplify:

1. $\dfrac{2\sqrt{-1}}{4 + 2\sqrt{-1}}$
2. $\dfrac{-2 + 4i}{-1 + 4i}$
3. $\dfrac{3 + \sqrt{-2}}{3 - \sqrt{-2}}$
4. $\dfrac{3}{1 - i\sqrt{3}}$
5. $\dfrac{1 - i}{2 - i}$
6. $\dfrac{8}{2 + \sqrt{-2}}$

Answers:

1. $\dfrac{2i + 1}{5}$
2. $\dfrac{18 + 4i}{17}$
3. $\dfrac{7 + 6i\sqrt{2}}{11}$
4. $\dfrac{3}{4} + \dfrac{3}{4}i\sqrt{3}$
5. $\dfrac{3 - i}{5}$
6. $\dfrac{8 - 4i\sqrt{2}}{3}$

CHAPTER 16

QUADRATIC EQUATIONS IN ONE VARIABLE

The degree of an equation in one variable is the exponent of the highest power to which the variable is raised in that equation. A second-degree equation in one variable is one in which the variable is raised to the second power. A second-degree equation is often called a QUADRATIC EQUATION. The word quadratic is derived from the Latin word quadratus, which means "squared." In a quadratic equation the term of highest degree is the squared term. For example, the following are quadratic equations:

$$x^2 + 3x + 4 = 0$$

$$3m + 4m^2 = 6$$

The terms of degree lower than the second may or may not be present. The possible terms of lower degree than the squared term in a quadratic equation are the first-degree term and the constant term. In the equation

$$3x^2 - 8x - 5 = 0$$

-5 is the coefficient of x^0. If we wished to emphasize the powers of x in this equation, we could write the equation in the form

$$3x^2 - 8x^1 - 5x^0 = 0$$

Examples of quadratic equations in which either the first-degree term or the constant term is missing are:

1. $4x^2 = 16$
2. $y^2 + 16y = 0$
3. $e^2 + 12 = 0$

GENERAL FORM OF A QUADRATIC EQUATION

Any quadratic equation can be arranged in the general form:

$$ax^2 + bx + c = 0$$

If it has more than three terms, some of them will be alike and can be combined, after which the final form will have at most three terms. For example,

$$2x^2 + 3 + 5x - 1 + x^2 = 4 - x^2 - 2x - 3$$

reduces to the simpler form

$$4x^2 + 7x + 1 = 0$$

In this form, it is easy to see that a, the coefficient of x^2, is 4; b, the coefficient of x, is 7; and c, the constant term, is 1.

Sometimes the coefficients of the terms of a quadratic appear as negative numbers, as follows:

$$2x^2 - 3x - 5 = 0$$

This equation can be rewritten in such a way that the connecting signs are all positive, as in the general form. This is illustrated as follows:

$$2x^2 + (-3)x + (-5) = 0$$

In this form, the value of a is seen to be 2, b is -3, and c is -5.

An equation of the form

$$x^2 + 2 = 0$$

has no x term. This can be considered as a case in which a is 1 (coefficient of x^2 understood to be 1), b is 0, and c is 2. For the purpose of emphasizing the values of a, b, and c with reference to the general form, this equation can be written

$$x^2 + 0x + 2 = 0$$

The coefficient of x^2 can never be 0; if it were 0, the equation would not be a quadratic. If the coefficients of x and x^0 are 0, then those terms do not normally appear. To say that the coefficient of x^0 is 0 is the same as saying that the constant term is 0 or is missing.

A ROOT of an equation in one variable is a value of the variable that satisfied the equation. Every equation in one variable, with constants as coefficients and positive integers as exponents, has as many roots as the exponent of the highest power. In other words, the number of roots is the same as the degree of the equation.

A fourth-degree equation has four roots, a cubic (third-degree) equation has three roots, a quadratic equation has two roots, and a linear equation has one root.

As an example, 6 and -1 are roots of the quadratic equation

$$x^2 - 5x - 6 = 0$$

This can be verified by substituting these values into the equation and noting that an identity results in each case.

Substituting $x = 6$ gives

$$\begin{aligned} 6^2 - 5(6) - 6 &= 0 \\ 36 - 36 &= 0 \\ 0 &= 0 \end{aligned}$$

Substituting $x = -1$ gives

$$\begin{aligned} (-1)^2 - 5(-1) - 6 &= 0 \\ 1 + 5 - 6 &= 0 \\ 6 - 6 &= 0 \\ 0 &= 0 \end{aligned}$$

Several methods of finding the roots of quadratic equations (SOLVING) are possible. The most common methods are solution by FACTORING and solution by the QUADRATIC FORMULA. Less commonly used methods of solution are accomplished by completing the square and by graphing.

SOLUTION BY FACTORING

The equation $x^2 - 36 = 0$ is a pure quadratic equation. There are two numbers which, when substituted for x, will satisfy the equation as follows:

$$\begin{aligned} (+6)^2 - 36 &= 0 \\ 36 - 36 &= 0 \end{aligned}$$

also

$$\begin{aligned} (-6)^2 - 36 &= 0 \\ 36 - 36 &= 0 \end{aligned}$$

Thus, +6 and -6 are roots of the equation

$$x^2 - 36 = 0$$

The most direct way to solve a pure quadratic (one in which no x term appears and the constant term is a perfect square) involves rewriting with the constant term in the right member, as follows:

$$x^2 = 36$$

Taking square roots on both sides, we have

$$x = \pm 6$$

The reason for expressing the solution as both plus and minus 6 is found in the fact that both +6 and -6, when squared, produce 36.

The equation

$$x^2 - 36 = 0$$

can also be solved by factoring, as follows:

$$\begin{aligned} x^2 - 36 &= 0 \\ (x + 6)(x - 6) &= 0 \end{aligned}$$

We now have the product of two factors equal to zero. According to the zero factor law, if a product is zero, then one or more of its factors is zero. Therefore, at least one of the factors must be zero, and it makes no difference which one. We are free to set first one factor and then the other factor equal to zero. In so doing we derive two solutions or roots of the equation.

If $x + 6$ is the factor whose value is 0, then we have

$$\begin{aligned} x + 6 &= 0 \\ x &= -6 \end{aligned}$$

If $x - 6$ is the zero factor, we have

$$\begin{aligned} x - 6 &= 0 \\ x &= 6 \end{aligned}$$

When a three-term quadratic is put into simplest form, it is customary to place all the terms on the left side of the equality sign with the squared term first, the first-degree term next, and the constant term last, as in

$$9x^2 - 2x + 7 = 0$$

If the trinomial in the left member is readily factorable, the equation can be solved quickly by separating the trinominal into factors. Consider the equation

$$3x^2 - x - 2 = 0$$

By factoring the trinominal, the equation becomes

$$(3x + 2)(x - 1) = 0$$

Once again we have two factors, the product of which is 0. This means that one or the other of them (or both) must have the value 0. If the zero factor is 3x + 2, we have

$$3x + 2 = 0$$
$$3x = -2$$
$$x = -\frac{2}{3}$$

If the zero factor is x - 1, we have

$$x - 1 = 0$$
$$x = 1$$

Substituting first x = 1 and then $x = -\frac{2}{3}$ in the original equation, we see that both roots satisfy it. Thus,

$$3(1)^2 - (1) - 2 = 0$$
$$3 - 1 - 2 = 0$$
$$0 = 0$$
$$3\left[-\frac{2}{3}\right]^2 - \left[-\frac{2}{3}\right] - 2 = 0$$
$$\frac{4}{3} + \frac{2}{3} - 2 = 0$$
$$0 = 0$$

In summation, when a quadratic may be readily factored, the process for finding its roots is as follows:

1. Arrange the equation in the order of the descending powers of the variable so that all the terms appear in the left member and zero appears in the right.
2. Factor the left member of the equation.
3. Set each factor containing the variable equal to zero and solve the resulting equations.
4. Check by substituting each of the derived roots in the original equation.

EXAMPLE: Solve the equation $x^2 - 4x = 12$ for x.

1. $x^2 - 4x - 12 = 0$
2. $(x - 6)(x + 2) = 0$
3. $x - 6 = 0 \qquad x + 2 = 0$
   $x = 6 \qquad x = -2$
4. $(6)^2 - 4(6) = 12 \quad (x = 6)$
   $36 - 24 = 12$
   $12 = 12$
   $(-2)^2 - 4\,(-2) = 12 \quad (x = -2)$
   $4 + 8 = 12$
   $12 = 12$

Practice problems. Solve the following equations by factoring:

1. $x^2 + 10x - 24 = 0$
2. $a^2 - a - 56 = 0$
3. $y^2 - 2y = 63$
4. $7y^2 - 19y - 6 = 0$
5. $m^2 - 4m = 96$

Answers:

1. $x = -12$
   $x = 2$
2. $a = 8$
   $a = -7$
3. $y = -7$
   $y = 9$
4. $y = 3$
   $y = -\frac{2}{7}$
5. $m = -8$
   $m = 12$

SOLUTION BY COMPLETING THE SQUARE

When a quadratic cannot be solved by factoring, or the factors are not readily seen, another method of finding the roots is needed. A method that may always be used for quadratics in one variable involves perfect square trinomials. These, we recall, are trinomials whose factors are identical. For example,

$$x^2 - 10x + 25 = (x - 5)(x - 5) = (x - 5)^2$$

Recall that in squaring a binomial, the third term of the resulting perfect square trinomial is always the square of the second term of the binomial. The coefficient of the middle term of the trinomial is always twice the second term of the binomial. For example, when (x + 4) is squared, we have

$$\begin{array}{l} x + 4 \\ \underline{x + 4} \\ x^2 + 4x \\ \underline{\quad\; + 4x + 16} \\ x^2 + 8x + 16 \end{array}$$

Hence if both the second- and first-degree terms of a perfect square trinomial are known, the third may be written by squaring one-half the coefficient of the first-degree term.

Essentially, in completing the square, certain quantities are added to one member and subtracted from the other, and the equation is so arranged that the left member is a perfect square trinomial. The square roots of both members may then be taken, and the subsequent equalities may be solved for the variable.

For example,

$$x^2 + 5x - \frac{11}{4} = 0$$

cannot be readily factored. To solve for x by completing the square, we proceed as follows:

1. Leave only the second- and first-degree terms in the left member.

$$x^2 + 5x = \frac{11}{4}$$

(If the coefficient of x^2 is not 1, divide through by the coefficient of x^2.)

2. Complete the square by adding to both members the square of half the coefficient of the x term. In this example, one-half of the coefficient of the x term is $\frac{5}{2}$, and the square of $\frac{5}{2}$ is $\frac{25}{4}$. Thus,

$$x^2 + 5x + \frac{25}{4} = \frac{11}{4} + \frac{25}{4}$$

3. Factor the left member and simplify the right member.

$$\left(x + \frac{5}{2}\right)^2 = 9$$

4. Take the square root of both members.

$$\sqrt{\left(x + \frac{5}{2}\right)^2} = \sqrt{9}$$

$$x + \frac{5}{2} = \pm 3$$

Remember that, in taking square roots on both sides of an equation, we must allow for the fact that two roots exist in every second-degree equation. Thus we designate both the plus and the minus root of 9 in this example.

5. Solve the resulting equations.

$$x + \frac{5}{2} = 3 \qquad\qquad x + \frac{5}{2} = -3$$

$$x = \frac{6}{2} - \frac{5}{2} \qquad\qquad x = -\frac{6}{2} - \frac{5}{2}$$

$$x = \frac{1}{2} \qquad\qquad x = -\frac{11}{2}$$

6. Check the results.

$$\left(\frac{1}{2}\right)^2 + \frac{5}{2} - \frac{11}{4} = 0$$

$$\frac{5}{2} - \frac{10}{4} = 0$$

$$0 = 0$$

$$\left(-\frac{11}{2}\right)^2 + (5)\left(-\frac{11}{2}\right) - \frac{11}{4} = 0$$

$$\frac{121}{4} - \frac{55}{2} - \frac{11}{4} = 0$$

$$\frac{110}{4} - \frac{55}{2} = 0$$

$$0 = 0$$

The process of completing the square may always be used to solve a quadratic equation. However, since this process may become complicated in more complex equations, a formula based on completing the square has been developed in which known quantities may be substituted in order to derive the roots of the quadratic equation. This formula is explained in the following paragraphs.

SOLUTION BY THE QUADRATIC FORMULA

The quadratic formula is derived by applying the process of completing the square to solve for x in the general form of the quadratic equation, $ax^2 + bx + c = 0$. Remember that the general form represents every possible quadratic equation. Thus, if we can solve this equation for x, the solution will be in terms of a, b, and c. To solve this equation for x by completing the square, we proceed as follows:

1. Subtract the constant term, c, from both members.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

2. Divide all terms by a so that the coefficient of the x^2 term becomes unity.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3. Add the square of one-half the coefficient of the x term, $\frac{b}{a}$, to both members.

$$\text{Square } \frac{b}{2a}\text{: } \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$$

$$\text{Add: } x + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

4. Factor the left member and simplify the right member.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

5. Take the square root of both members.

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

6. Solve for x.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, we have solved the equation representing every quadratic for its unknown in terms of its constants a, b, and c. Hence, in a given quadratic we need only substitute in the expression

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

the values of a, b, and c, as they appear in the particular equation, to derive the roots of that equation. This expression is called the QUADRATIC FORMULA. The general quadratic equation, $ax^2 + bx + c = 0$, and the quadratic formula should be memorized. Then, when a quadratic cannot be solved quickly by factoring, it may be solved at once by the formula.

EXAMPLE: Use the quadratic formula to solve the equation

$$x^2 + 30 - 11x = 0.$$

SOLUTION:

1. Set up the equation in standard form.

$$x^2 - 11x + 30 = 0$$

Then a (coefficient of x^2) = 1
b (coefficient of x) = -11
c (the constant term) = 30

2. Substituting,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-11) \pm \sqrt{(-11)^2 - 4(1)(30)}}{2(1)}$$

$$= \frac{11 \pm \sqrt{121 - 120}}{2}$$

$$= \frac{11 \pm 1}{2} = 6 \text{ or } 5$$

3. Checking:

When

$$x = 6,$$

$$(6)^2 - 11(6) + 30 = 0$$

$$36 - 66 + 30 = 0$$

$$0 = 0$$

When

$$x = 5,$$

$$(5)^2 - 11(5) + 30 = 0$$

$$25 - 55 + 30 = 0$$

$$0 = 0$$

EXAMPLE: Find the roots of

$$2x^2 - 3x - 1 = 0$$

Here, a = 2, b = -3, and c = -1.

Substituting into the quadratic formula gives

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9 + 8}}{4}$$

$$= \frac{3 \pm \sqrt{17}}{4}$$

The two roots are

$$x = \frac{3}{4} + \frac{1}{4}\sqrt{17} \text{ and } x = \frac{3}{4} - \frac{1}{4}\sqrt{17}$$

These roots are irrational numbers, since the radicals cannot be removed.

If the decimal values of the roots are desired, the value of the square root of 17 can be taken from appendix I of this course. Substituting $\sqrt{17} = 4.1231$ and simplifying gives

$$x_1 = \frac{3 + 4.1231}{4} \quad \text{and} \quad x_2 = \frac{3 - 4.1231}{4}$$

$$x_1 = \frac{7.1231}{4} \qquad x_2 = \frac{-1.1231}{4}$$

$$x_1 = 1.781 \qquad x_2 = -0.281$$

In decimal form, the roots of $2x^2 - 3x - 1 = 0$ to the nearest tenth are 1.8 and -0.3.

Notice that the subscripts, 1 and 2, are used to distinguish between the two roots of the equation. The three roots of a cubic equation in x might be designated x_1, x_2, and x_3. Sometimes the letter r is used for root. Using r, the roots of a cubic equation could be labeled r_1, r_2, and r_3.

Checking:

When $$x_1 = \frac{3 + \sqrt{17}}{4}$$

$$2x^2 - 3x - 1 = 0$$

then

$$2\left(\frac{3 + \sqrt{17}}{4}\right)^2 - 3\left(\frac{3 + \sqrt{17}}{4}\right) - 1 = 0$$

$$\frac{(3 + \sqrt{17})^2}{8} - \frac{9 + 3\sqrt{17}}{4} - 1 = 0$$

$$\frac{9 + 6\sqrt{17} + 17 - 18 - 6\sqrt{17} - 8}{8} = 0$$

$$0 = 0$$

When

$$x_2 = \frac{3 - \sqrt{17}}{4}$$

then

$$2\left(\frac{3 - \sqrt{17}}{4}\right)^2 - 3\left(\frac{3 - \sqrt{17}}{4}\right) - 1 = 0$$

$$\frac{9 - 6\sqrt{17} + 17}{8} - \frac{9 - 3\sqrt{17}}{4} - 1 = 0$$

Multiplying both members of the equation by 8, the LCD, we have

$$8\left(\frac{9 - 6\sqrt{17} + 17}{8}\right) - 8\left(\frac{9 - 3\sqrt{17}}{4}\right) - 8(1) = 0$$

$$9 - 6\sqrt{17} + 17 - 2(9 - 3\sqrt{17}) - 8 = 0$$

$$9 - 6\sqrt{17} + 17 - 18 + 6\sqrt{17} - 8 = 0$$

$$0 = 0$$

Practice problems. Use the quadratic formula to find the roots of the following equations:

1. $3x^2 - 20 - 7x = 0$
2. $4x^2 - 3x - 5 = 0$
3. $15x^2 - 22x - 5 = 0$
4. $x^2 + 7x = 8$

Answers:

1. $x_1 = 4$

   $x_2 = -\frac{5}{3}$
2. $x_1 = \frac{3 + \sqrt{89}}{8}$

   $x_2 = \frac{3 - \sqrt{89}}{8}$
3. $x_1 = \frac{5}{3}$

   $x_2 = -\frac{1}{5}$
4. $x_1 = 1$

   $x_2 = -8$

GRAPHICAL SOLUTION

A fourth method of solving a quadratic equation is by means of graphing. In graphing linear equations using both axes as reference, we recall that an independent variable, x, and a dependent variable, y, were needed. The coordinates of points on the graph of the equation were designated (x, y).

Since the quadratics we are considering contain only one variable, as in the equation

$$x^2 - 8x + 12 = 0$$

we cannot plot values for the equations in the present form using both x and y axes. A dependent variable, y, is necessary.

If we think of the expression

$$x^2 - 8x + 12$$

as a function, then this function can be considered to have many possible numerical values, depending on what value we assign to x. The particular value or values of x which cause the value of the function to be 0 are solutions for the equation

$$x^2 - 8x + 12 = 0$$

For convenience, we may choose to let y represent the function

$$x^2 - 8x + 12$$

If numerical values are now assigned to x, the corresponding values of y may be calculated. When these pairs of corresponding values of x and y are tabulated, the resulting table provides the information necessary for plotting a graph of the function.

EXAMPLE: Graph the equation

$$x^2 + 2x - 8 = 0$$

and from the graph write the roots of the equation.

SOLUTION:

1. Let $y = x^2 + 2x - 8$.

2. Make a table of the y values corresponding to the value assigned x, as shown in table 16-1.

Table 16-1.—Tabulation of x and y values for the function $y = x^2 + 2x - 8$.

if x = -----	-5	-4	-3	-2	-1	0	1	2	3
then y ---	7	0	-5	-8	-9	-8	-5	0	7

3. Plot the pairs of x and y values that appear in the table as coordinates of points on a rectangular coordinate system as in figure 16-1 (A).

4. Draw a smooth curve through these points, as shown in figure 16-1 (B).

Notice that this curve crosses the X axis in two places. We also recall that, for any point on the X axis, the y coordinate is zero. Thus, in the figure we see that when y is zero, x is -4 or +2. When y is zero, furthermore, we have the original equation,

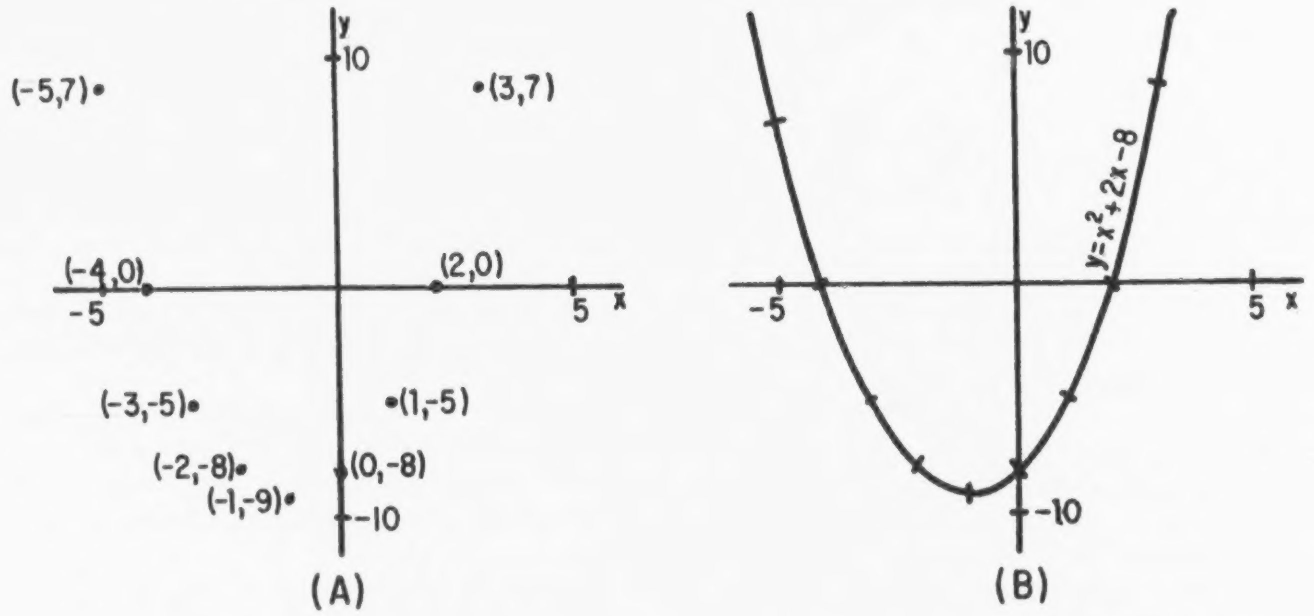


Figure 16-1.—Graph of the equation $y = x^2 + 2x - 8$. (A) Points plotted; (B) curve drawn through plotted points.

$$x^2 + 2x - 8 = 0$$

Thus, the values of x at these points where the graph of the equation crosses the X axis ($x = -4$ or $+2$) are solutions to the original equation. We may check these results by solving the equation algebraically. Thus,

$$x^2 + 2x - 8 = 0$$

$$(x + 4)(x - 2) = 0$$

$$x_1 + 4 = 0 \qquad x_2 - 2 = 0$$

$$x_1 = -4 \qquad x_2 = 2$$

Check:

$$(-4)^2 + 2(-4) - 8 = 0 \qquad (2)^2 + 2(2) - 8 = 0$$

$$16 - 8 - 8 = 0 \qquad 4 + 4 - 8 = 0$$

$$0 = 0 \qquad 0 = 0$$

The curve in figure 16-1 (B) is called a PARABOLA. Every quadratic of the form $ax^2 + bx + c = y$ will have a graph of this general shape. The curve will open downward if a is negative, and upward if a is positive.

Graphing provides a fourth method of finding the roots of a quadratic in one variable. When the equation is graphed, the roots will be the X intercepts (those values of x where the curve crosses the X axis). The X intercepts are the points at which y is 0.

Practice problems. Graph the following quadratic equations and read the roots of each equation from its graph

1. $x^2 - 4x - 8 = 0$
2. $6x - 5 - x^2 = 0$

Answers:

1. See figure 16-2. $x = 5.5$; $x = -1.5$
2. See figure 16-3. $x = 1$; $x = 5$

MAXIMUM AND MINIMUM POINTS

It will be seen from the graphs of quadratics in one variable that a parabola has a maximum or minimum value, depending on whether the curve opens upward or downward. Thus, when a is negative the curve passes through a maximum value; and when a is positive, the curve passes through a minimum value. Often these maximum or minimum values comprise the only information needed for a particular problem.

In higher mathematics it can be shown that the X coordinate, or abscissa, of the maximum or minimum value is

$$x = \frac{-b}{2a}$$

In other words, if we divide minus the coefficient of the x term by twice the coefficient of the x^2 term, we have the X coordinate of the maximum or minimum point. If we substitute this value for x in the original equation, the result is the Y value or ordinate, which corresponds to the X value.

For example, we know that the graph of the equation

$$x^2 + 2x - 8 = y$$

passes through a minimum value because a is positive. To find the coordinates of the point where the parabola has its minimum value, we note that $a = 1$, $b = 2$, $c = -8$. From the rule given above, the X value of the minimum point is

$$x = \frac{-b}{2a}$$

$$x = \frac{-(2)}{2(1)}$$

$$x = -1$$

Substituting this value for x in the original equation, we have the value of the Y coordinate of the minimum point. Thus,

$$(-1)^2 + 2(-1) - 8 = y$$

$$1 - 2 - 8 = y$$

$$-9 = y$$

The minimum point is $(-1, -9)$. From the graph in figure 16-1 (A), we see that these coordinates are correct. Thus, we can quickly and easily find the coordinates of the minimum or maximum point for any quadratic of the form $ax^2 + bx + c = 0$.

Practice problems. Without graphing, find the coordinates of the maximum or minimum points for the following equations and state whether they are maximum or minimum.

1. $2x^2 - 5x + 2 = 0$
2. $68 - 3x - x^2 = 0$
3. $3 + 7x - 6x^2 = 0$
4. $24x^2 - 14x = 3$

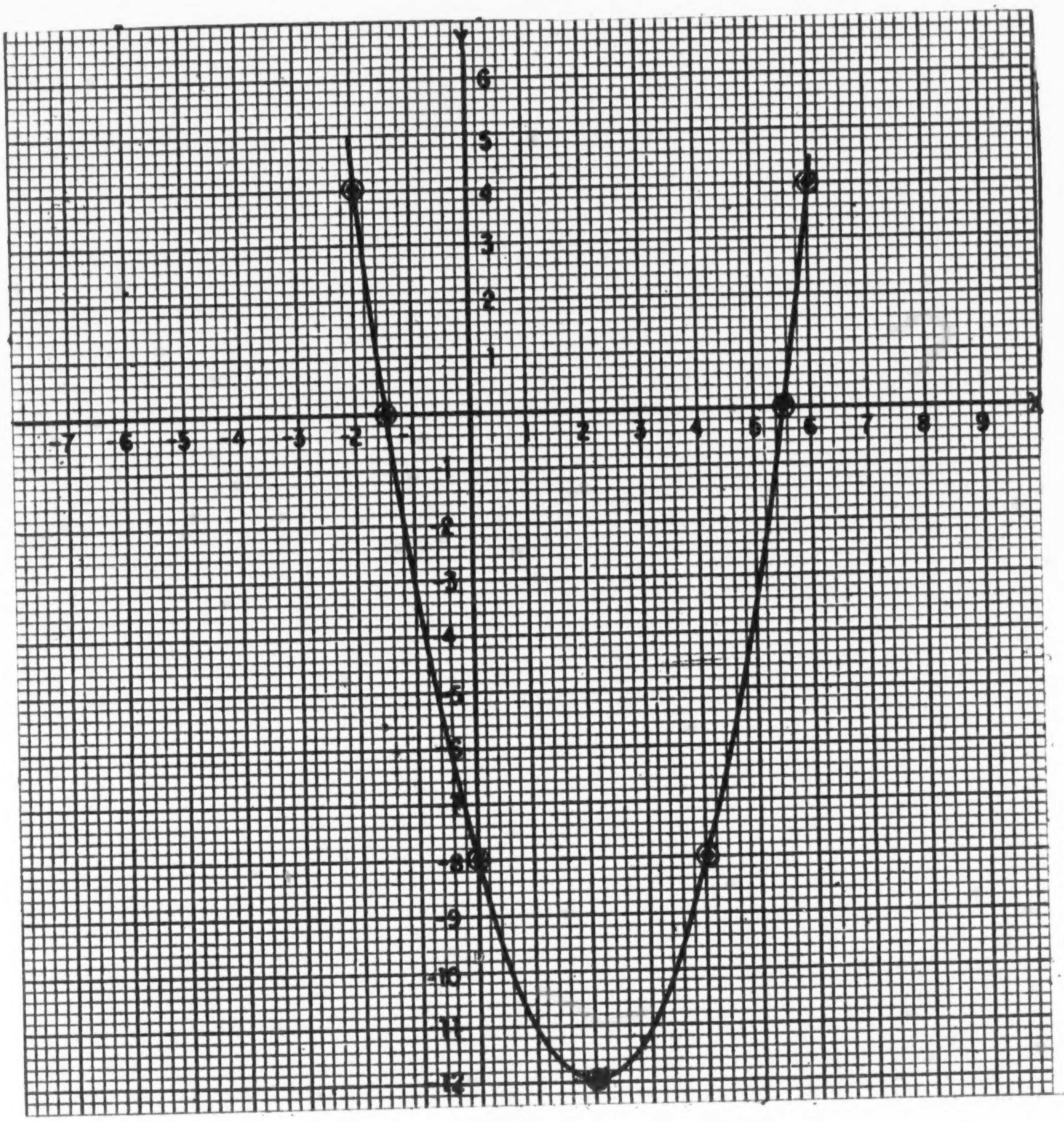


Figure 16-2.—Graph of $x^2 - 4x - 8 = 0$.

Answers:

1. $x = \frac{5}{4}$ Minimum

$y = -\frac{9}{8}$

2. $x = -\frac{3}{2}$ Maximum

$y = \frac{281}{4}$

3. $x = \frac{7}{12}$ Maximum

$y = \frac{121}{24}$

4. $x = \frac{7}{24}$ Minimum

$y = -\frac{121}{24}$

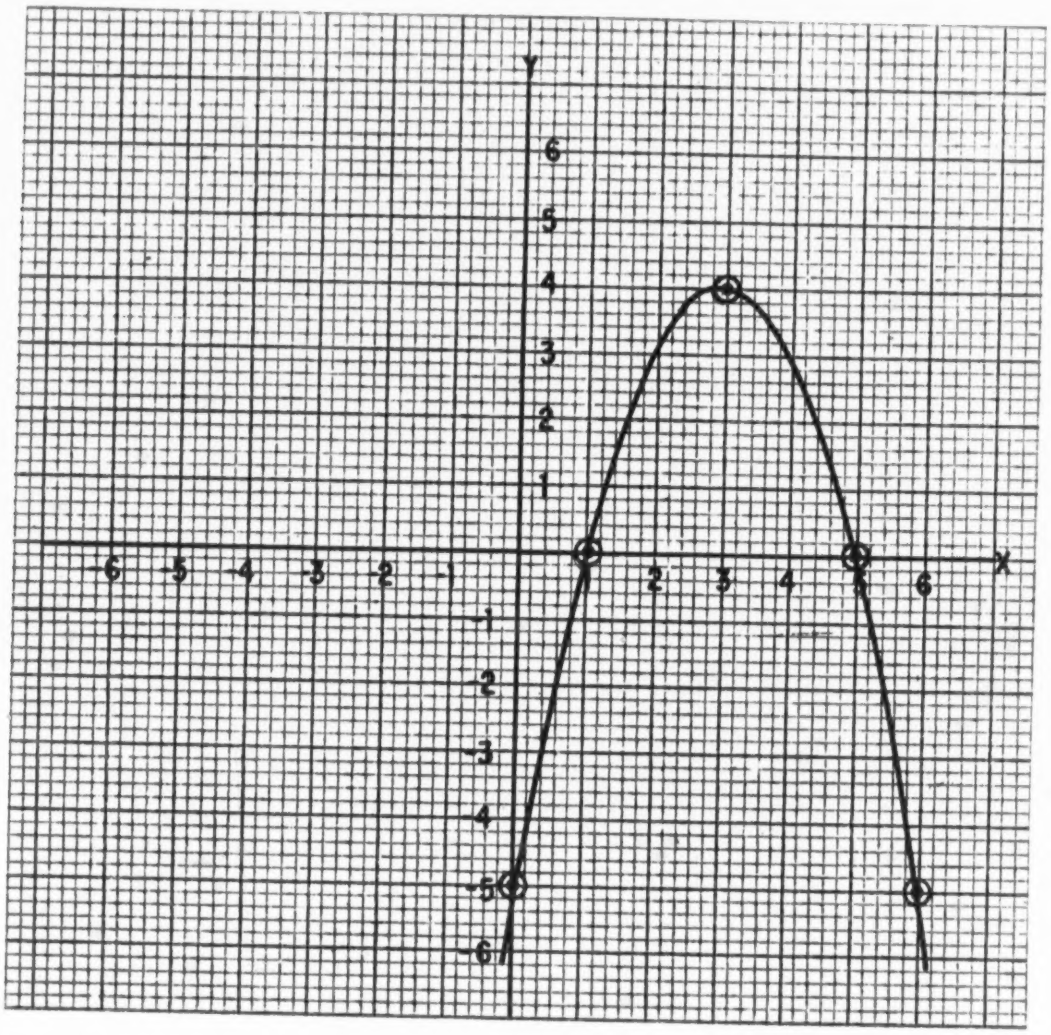


Figure 16-3.—Graph of $6x - 5 - x^2 = 0$.

THE DISCRIMINANT

The roots of a quadratic equation may be classified in accordance with the following criteria:

1. Real or imaginary.
2. Rational or irrational.
3. Equal or unequal.

The task of discriminating among these possible characteristics to find the nature of the roots is best accomplished with the aid of the quadratic formula. The part of the quadratic formula which is used is called the DISCRIMINANT.

If the roots of a quadratic are denoted by the symbols r_1 and r_2, then the following relations may be stated:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We can show that the character of the roots is dependent upon the form taken by the expression

$$b^2 - 4ac$$

which is the quantity under the radical in the formula. This expression is the DISCRIMINANT of a quadratic equation.

IMAGINARY ROOTS

Since there is a radical in each root, there is a possibility that the roots could be imaginary. They are imaginary when the number under the radical in the quadratic formula is negative (less than 0). In other words, when the value of the discriminant is less than 0, the roots are imaginary.

EXAMPLE:

$$x^2 + x + 1 = 0$$

$$a = 1,\ b = 1,\ c = 1$$

$$\begin{aligned} b^2 - 4ac &= (1)^2 - 4(1)(1) \\ &= 1 - 4 \\ &= -3 \end{aligned}$$

Thus, without further work, we know that the roots are imaginary.

CHECK: The roots are

$$r_1 = \frac{-1 + \sqrt{-3}}{2} \qquad r_2 = \frac{-1 - \sqrt{-3}}{2}$$

$$r_1 = -\frac{1}{2} + \frac{i\ \sqrt{3}}{2} \qquad r_2 = -\frac{1}{2} - \frac{i\ \sqrt{3}}{2}$$

We recognize both of these numbers as being imaginary.

We may also conclude that when one root is imaginary the other will also be imaginary. This is because the pairs of imaginary roots are always conjugate complex numbers. If one root is of the form a + ib, then a - ib is also a root. Knowing that imaginary roots always occur in pairs, we can conclude that a quadratic equation always has either two imaginary roots or two real roots.

Practice problems. Using the discriminant, state whether the roots of the following equations are real or imaginary:

1. $x^2 - 6x - 16 = 0$
2. $x^2 - 6x = -12$
3. $3x^2 - 10x + 50 = 0$
4. $6x^2 + x = 1$

Answers:

1. Real
2. Imaginary
3. Imaginary
4. Real

EQUAL OR DOUBLE ROOTS

If the discriminant $b^2 - 4ac$ equals zero, the radical in the quadratic formula becomes zero. In this case the roots are equal; such roots are sometimes called double roots.

Consider the equation

$$9x^2 + 12x + 4 = 0$$

Comparing with the general quadratic, we notice that

$$a = 9,\ b = 12, \text{ and } c = 4$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= 12^2 - 4(9)(4) \\ &= 144 - 144 \\ &= 0 \end{aligned}$$

Therefore, the roots are equal.

CHECK: From the formula

$$r_1 = \frac{-12 + 0}{2(9)} \qquad r_2 = \frac{-12 - 0}{2(9)}$$

$$r_1 = -\frac{2}{3} \qquad r_2 = -\frac{2}{3}$$

The equality of the roots is thus verified.

The roots can be equal only if the trinomial is a perfect square. Its factors are equal. Factoring the trinomial in

$$9x^2 + 12x + 4 = 0$$

we see that

$$(3x + 2)^2 = 0$$

Since the factor 3x + 2 is squared, we actually have

$$3x + 2 = 0$$

twice, and we have

$$x = -\frac{2}{3}$$

twice.

The fact that the same root must be counted twice explains the use of the term "double root." A double root of a quadratic equation is always rational because a double root can occur only when the radical vanishes.

REAL AND UNEQUAL ROOTS

When the discriminant is positive, the roots must be real. Also they must be unequal since equal roots occur only when the discriminant is zero.

Rational Roots

If the discriminant is a perfect square, the roots are rational. For example, consider the equation

$$3x^2 - x - 2 = 0$$

in which

$$2 = 3,\ b = -1,\ \text{and}\ c = -2$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(3)(-2) \\ &= 1 + 24 \\ &= 25 \end{aligned}$$

We see that the discriminant, 25, is a perfect square. The perfect square indicates that the radical in the quadratic formula can be removed, that the roots of the equation are rational, and that the trinomial can be factored. In other words, when we evaluate the discriminant and find it to be a perfect square, we know that the trinomial can be factored.

Thus,

$$\begin{aligned} 3x^2 - x - 2 &= 0 \\ (3x + 2)(x - 1) &= 0 \end{aligned}$$

from which

$$\begin{aligned} 3x + 2 &= 0 \qquad & x - 1 &= 0 \\ x &= -\frac{2}{3} & x &= 1 \end{aligned}$$

We see that the information derived from the discriminant is correct. The roots are real, unequal, and rational.

Irrational Roots

If the discriminant is not a perfect square, the radical cannot be removed and the roots are irrational.

Consider the equation

$$2x^2 - 4x + 1 = 0$$

in which

$$a = 2,\ b = -4,\ \text{and}\ c = 1.$$

The discriminant is

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(2)(1) \\ &= 16 - 8 \\ &= 8 \end{aligned}$$

This discriminant is positive and not a perfect square. Thus the roots are real, unequal, and irrational.

To check the correctness of this information, we derive the roots by means of the formula. Thus,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{4 \pm \sqrt{8}}{4} \\ &= \frac{2 \pm \sqrt{2}}{2} \end{aligned}$$

$$x = 1 + \frac{\sqrt{2}}{2} \text{ or } x = 1 - \frac{\sqrt{2}}{2}$$

This verifies the conclusions reached in evaluating the discriminant. When the discriminant is a positive number, not a perfect square, it is useless to attempt to factor the trinomial. The formula is needed to find the roots. They will be real, unequal, and irrational.

SUMMARY

The foregoing information concerning the discriminant may be summed up in the following four rules:

1. If $b^2 - 4ac$ is a perfect square or zero, the roots are rational; otherwise they are irrational.
2. If $b^2 - 4ac$ is negative (less than zero), the roots are imaginary.
3. If $b^2 - 4ac$ is zero, the roots are real, equal, and rational.
4. If $b^2 - 4ac$ is greater than zero, the roots are real and unequal.

Practice problems. Determine the character of the roots of each of the following equations:

1. $x^2 - 7x + 12 = 0$
2. $9x^2 - 6x + 1 = 0$
3. $2x^2 - x + 1 = 0$
4. $2x - 2x^2 + 6 = 0$

Answers:

1. Real, unequal, rational
2. Real, equal, rational
3. Imaginary
4. Real, unequal, irrational

GRAPHICAL INTERPRETATION OF ROOTS

When a quadratic is set equal to y and the resulting equation is graphed, the graph will reveal the character of the roots, but it may not reveal whether the roots are rational or irrational.

Consider the following equations:

1. $x^2 + 6x - 3 = y$
2. $x^2 + 6x + 9 = y$
3. $x^2 + 6x + 13 = y$

The graphs representing these equations are shown in figure 16-4.

We recall that the roots of the equation are the values of x at those points where y is zero. Y is zero on the graph anywhere along the X axis. Thus, the roots of the equation are the positions where the graph crosses the X axis. In parabola No. 1 (fig. 16-4) we see immediately that there are two roots to the equation and that they are unequal. These roots appear to be -6.5 and 0.5. Algebraically, we find them to be the irrational numbers

$$-3 + 2\sqrt{3} \text{ and } -3 - 2\sqrt{3}.$$

For equation No. 2 (fig. 16-4), the parabola just touches the X axis at x = -3. This means that both roots of the equation are the same—that is, the root is a double root. At the point where the parabola touches the X axis, the two roots of the quadratic equation have moved together and the two points of intersection of the parabola and the X axis are coincident. The quantity -3 as a double root agrees with the algebraic solution.

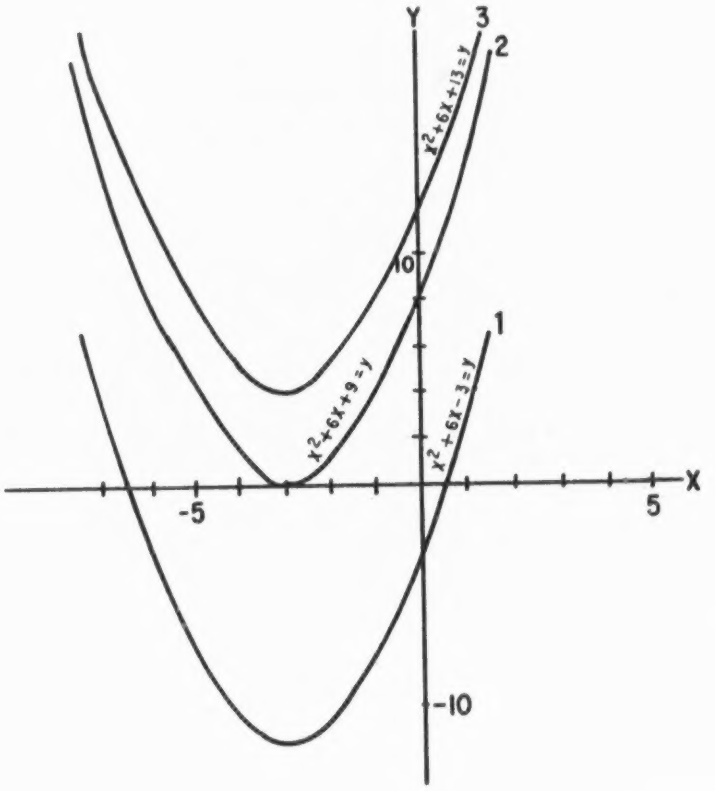


Figure 16-4.—Graphical interpretation of roots.

When the equation No. 3 (fig. 16-4) is solved algebraically, we see that the roots are -3 + 2i and -3 - 2i. Thus they are imaginary. Parabola No. 3 does not cross the X axis. When this situation occurs, imaginary roots are implied. Only equations having real roots will have graphs that cross or touch the X axis. Thus we may determine from the graph of an equation whether the roots are real or imaginary.

VERBAL PROBLEMS INVOLVING QUADRATIC EQUATIONS

Many practical problems give rise to quadratic equations. In such problems it often happens that one of the roots will have no meaning. We must select the root that satisfies the conditions of the problem.

Consider the following example: The length of a plot of ground exceeds its width by 7 ft and the area of the plot is 120 sq ft. What are the dimensions?

SOLUTION:

Let x = length
y = width

then

$$x - y = 7 \qquad (1)$$

and

$$xy = 120 \qquad (2)$$

Solving (1) for y, $y = x - 7$

Substituting $(x - 7$ for y in (2)

$$x(x - 7) = 120$$

Therefore

$$x^2 - 7x - 120 = 0$$

$$(x - 15)(x + 8) = 0$$

$$x = 15, \qquad x = -8$$

Thus, length = +15 or -8.

But the length obviously cannot be a negative value. Therefore, we reject -8 as a value for x and use only the positive value, +15. Then from equation (1),

$$15 - y = 7$$

$$y = 8$$

Length = 15, Width = 8

Practice problems. Solve the following problems by forming quadratic equations:

1. A rectangular plot is 8 yd by 24 yd. If the length and width are increased by the same amount, the area is increased by 144 sq yd. How much is each dimension increased?

2. Two cars travel at uniform rates of speed over the same route a distance of 180 mi. One goes 5 mph slower than the other and takes 1/2 hr longer to make the run. How fast does each car travel?

Answers:

1. Length and width are each increased by 4 yd.
2. Faster car: 45 mph.
   Slower car: 40 mph.

CHAPTER 17

PLANE FIGURES

The discussion of lines and planes in chapter 1 of this course was limited to their consideration as examples of sets. The present chapter is concerned with lines, angles, and areas as found in various plane (flat) geometric figures.

LINES

In the strictly mathematical sense, the term "line segment" should be used whenever we refer to the straight line joining some point A to some other point B. However, since the straight lines comprising geometric figures have clearly designated end points, we may simplify our terminology. Throughout the remaining chapters of this course, the general term "line" is used to designate straight line segments, unless stated otherwise.

TYPES OF LINES

The two basic types of lines in geometry are straight lines and curved lines. A curved line joining points A and B is designated as "curve AB." (See fig. 17-1.) If curve AB is an arc of a circle, it may be designated as "arc AB."

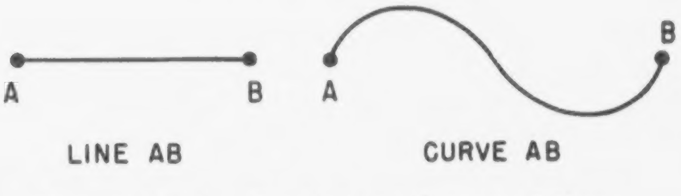


Figure 17-1.—Straight and curved lines.

The term "broken line" in mathematics means a series of two or more straight segments connected end-to-end but not running in the same direction. In mathematics, a series of short, straight segments with breaks between them, which would form a single straight line if joined end-to-end, is a DASHED LINE. (See fig. 17-2.)

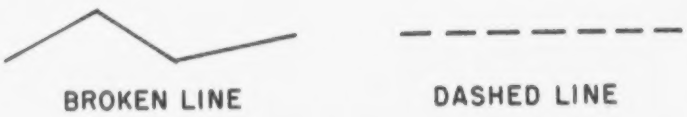


Figure 17-2.—Broken and dashed lines.

ORIENTATION

Straight lines may be classified in terms of their orientation to the observer's horizon or in terms of their orientation to each other. For example, lines in the same plane which run beside each other without meeting at any point, no matter how far they are extended, are PARALLEL. (See fig. 17-3 (A).) Lines in the same plane which are not parallel are OBLIQUE. Oblique lines meet to form angles (discussed in the following section). If two oblique lines cross or meet in such a way as to form four equal angles, as in figure 17-3 (B), the lines are PERPENDICULAR. This definition includes the case in which only one angle is formed, such as angle AEC in figure 17-3 (C). By extending line AE to form line AD, and extending CE to form CB, four equal angles (AEC, CED, DEB, and BEA) are formed.

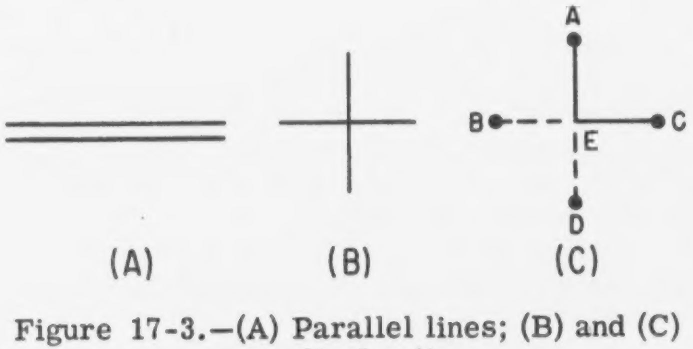


Figure 17-3.—(A) Parallel lines; (B) and (C) perpendicular lines.

Lines parallel to the horizon are HORIZONTAL. Lines perpendicular to the horizon are VERTICAL.

ANGLES

Lines which meet or cross each other are said to INTERSECT. Angles are formed when two straight lines intersect. The two lines which form an angle are its SIDES, and the point where the sides intersect is the VERTEX. In figure 17-4, the sides of the angles are AV and BV, and the vertex is V in each case. Figure 17-4 (A) is an ACUTE angle; (B) is an OBTUSE angle.

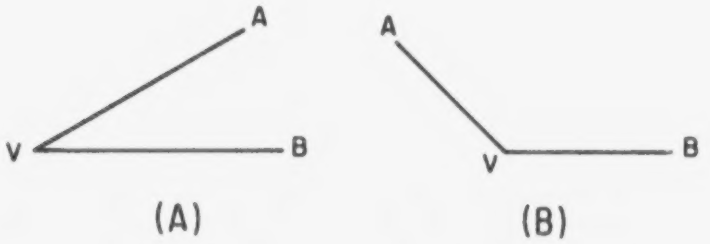


Figure 17-4.—(A) Acute angle; (B) obtuse angle.

CLASSIFICATION BY SIZE

When the sides of an angle are perpendicular to each other, the angle is a RIGHT angle. This term is related to the Latin word "rectus," which may be translated "erect" or "upright." Thus, if one side of a right angle is horizontal, the other side is erect or upright.

The size of an angle refers to the amount of separation between its sides, and the unit of angular size is the angular DEGREE. A right angle contains 90 degrees, abbreviated 90°. An angle smaller than a right angle is acute; an angle larger than a right angle is obtuse. Therefore, acute angles are angles of less than 90°, and obtuse angles are angles between 90° and 180°.

If side AV in figure 17-5 (A) is moved downward, the size of the obtuse angle AVB is increased. If side AV is moved so far that it coincides with (lies on top of) CV as in figure 17-5 (B), an angle is formed which is equal to the sum of two right angles. The special angle thus formed (AVB) is a straight angle, so called because it is visually indistinguishable from a straight line.

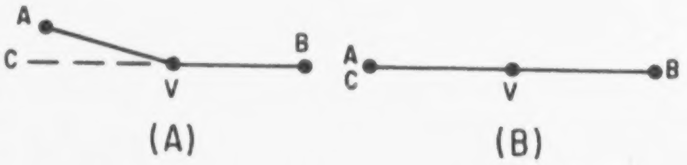


Figure 17-5.—(A) Large obtuse angle; (B) straight angle.

GEOMETRIC RELATIONSHIPS

Angles are often classified by their relationship to other angles or to other parts of a geometric figure. For example, angles 1 and 3 in figure 17-6 are VERTICAL angles, so called because they share a common vertex. Angles 2 and 4 are also vertical angles. Lines which cross, as in figure 17-6, always form two pairs of vertical angles, and the vertical angles thus formed are equal in pairs; that is, angle 1 equals angle 3, and angle 2 equals angle 4.

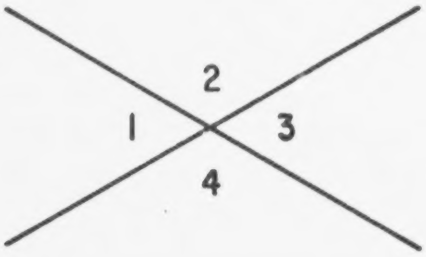


Figure 17-6.—Vertical angles.

Angles 1 and 2 in figure 17-6 are ADJACENT angles. Other pairs of adjacent angles in figure 17-6 are 2 and 3, 3 and 4, and 1 and 4. In the sense used here, adjacent means side by side, not merely close together or touching. For example, angles 1 and 3 are not adjacent angles even though they touch each other.

COMPLEMENTS AND SUPPLEMENTS

Two angles whose sum is 90° are complementary. For example, a 60° angle is the complement of a 30° angle, and conversely. "Conversely" is a mathematical word meaning "vice versa." Two angles whose sum is 180° are supplementary. For example, a 100° angle is the supplement of an 80° angle, and conversely.

Practice problems.

1. Describe the angle which is the complement of an acute angle.

2. Describe the angle which is the supplement of a right angle.

3. If two equal angles are complementary, each contains how many degrees?

4. Find the size of an angle which is twice as large as its own complement.

(Hint: If x is the angle, then 90° - x is its complement.)

Answers:

1. Acute
2. Right
3. 45°
4. 60°

GEOMETRIC FIGURES

The discussion of geometric figures in this chapter is limited to polygons and circles. A POLYGON is a plane closed figure, the sides of which are all straight lines. Among the polygons discussed are triangles, parallelograms, and trapezoids.

TRIANGLES

A triangle is a polygon which has three sides and three angles. In general, any polygon has as many angles as it has sides, and conversely.

Parts of a Triangle

Each of the three angles of a triangle is a VERTEX; therefore, every triangle has three vertices. The three straight lines joining the vertices are the SIDES (sometimes called legs), and the side upon which the triangle rests is its BASE, often designated by the letter b. This definition assumes that the standard position of a triangle drawn for general discussion is as shown in figure 17-7, in which the triangle is lying on one of its sides. The vertex opposite the base is the highest point of a triangle in standard position, and is thus called the APEX.

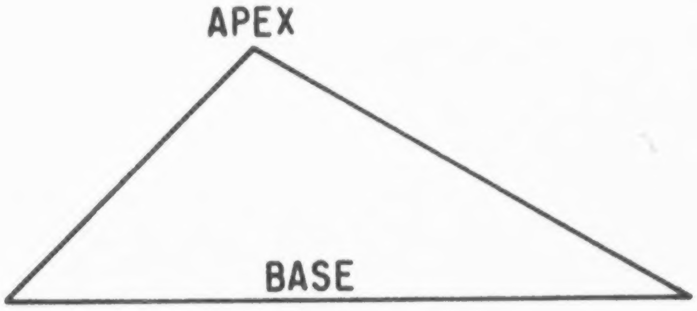


Figure 17-7.—Triangle in standard position.

A straight line perpendicular to the base of a triangle, joining the base to the apex, is the ALTITUDE, often designated by the letter a. The altitude is sometimes referred to as the height, and is then designated by the letter h. Figure 17-8 (B) shows that the apex may not be situated directly above the base. In this case, the base must be extended, as shown by the dashed line, in order to drop a perpendicular from the apex to the base. Mathematicians often use the term "drop a perpendicular." The meaning is the same as "draw a straight, perpendicular line."

In general, the geometrical term "distance from a point to a line" means the length of a perpendicular dropped from the point to the line. Many straight lines could be drawn from a line to a point not on the line, but the shortest of these is the one we use in measuring the

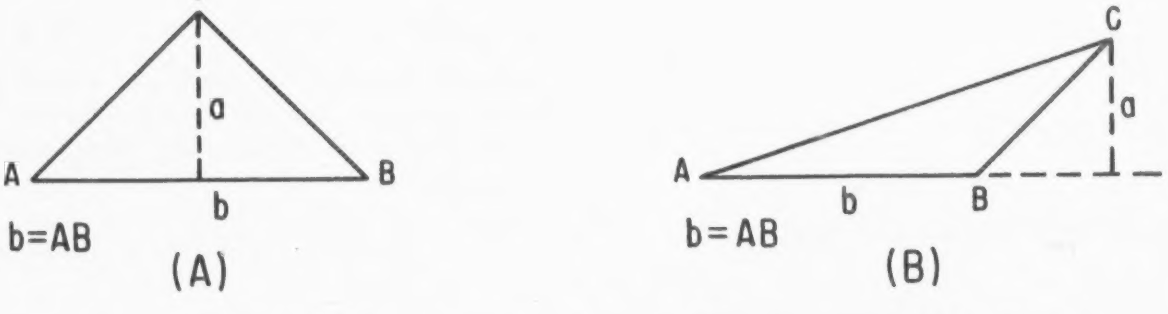


Figure 17-8.—(A) Interior altitude line; (B) exterior altitude line.

distance from the point to the line. The shortest one is perpendicular to the line.

Perimeter and Area

The PERIMETER of a triangle is the sum of the lengths of its sides. In less precise terms, this is sometimes stated as "the distance around the triangle." If the three sides are labeled a, b, and c, the perimeter P can be found by the following formula:

$$P = a + b + c$$

The area of a triangle is the space bounded (enclosed) by its sides. The formula for the area can be found by using a triangle which is part of a rectangle. In figure 17-9, triangle ABC is one-half of the rectangle. Since the area of the rectangle is a times b (that is, ab), the area of the triangle is given by the following formula:

$$\text{Area} = \frac{1}{2}\,ab$$

Written in terms of h, representing height, the formula is:

$$A = \frac{1}{2}\,bh$$

This formula is valid for every triangle, including those with no two sides perpendicular.

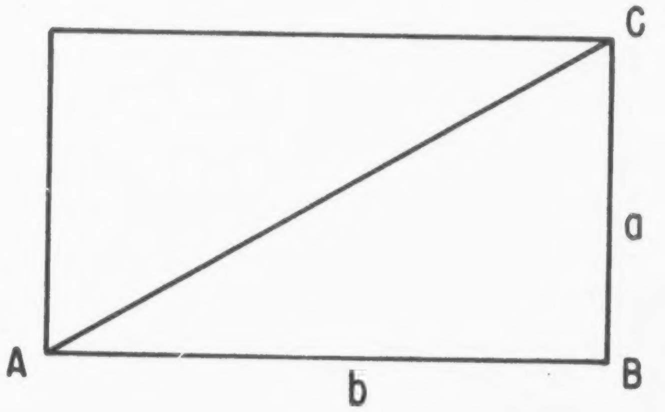


Figure 17-9.—Area of a triangle.

Practice problems. Find the perimeter and area of each of the triangles in figure 17-10.

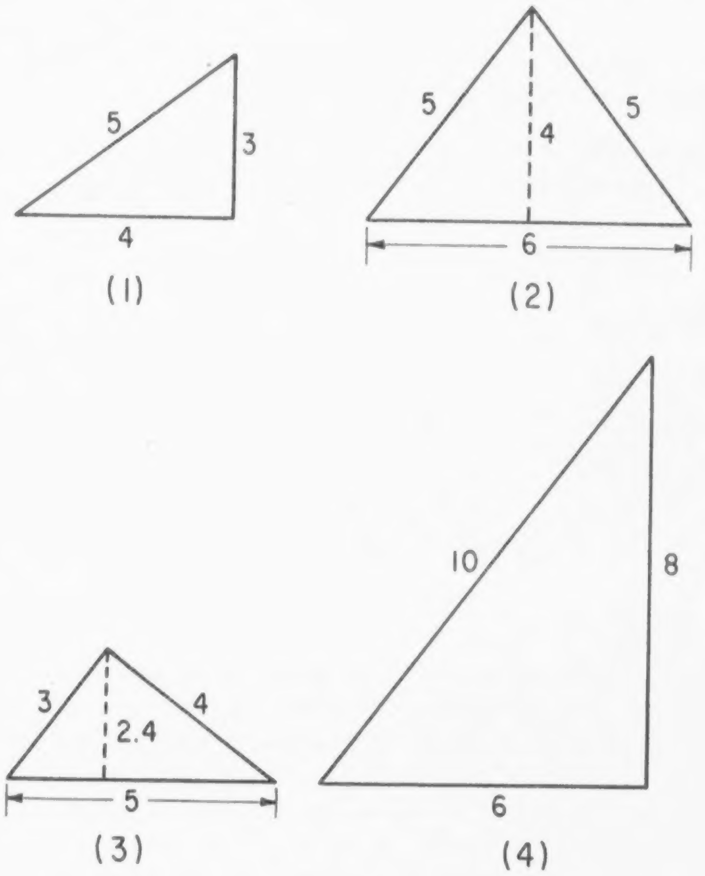


Figure 17-10.—Perimeters and areas of triangles.

Answers:

1. P = 12 units
   A = 6 square units
2. P = 16 units
   A = 12 square units
3. P = 12 units
   A = 6 square units
4. P = 24 units
   A = 24 square units

CAUTION: The concept of area is meaningless if the units of the multiplied dimensions are not the same. For example, if the base of a triangle is 2 feet long and the altitude is 6 inches long, the area might be carelessly stated as $\frac{1}{2}$ (6) (2). However, the units must be considered in order to decide whether the answer is in square feet or square inches. When the units are considered, we realize that the correct answer is

$$\frac{1}{2}\ (6\text{ in.})\ (24\text{ in.}) = 72\text{ sq in.}$$

$$\frac{1}{2}\left(\frac{1}{2}\text{ ft}\right)(2\text{ ft}) = \frac{1}{2}\text{ sq ft}$$

Special Triangles

The classification of triangles depends upon their special characteristics, if any. For example, a triangle may have all three of its sides equal in length; it may have two equal sides and a third side which is longer or shorter than the other two; it may contain a right angle or an obtuse angle. If it has none of these special characteristics, it is a SCALENE triangle. A scalene triangle has no two of its sides equal and no two of its angles equal.

RIGHT TRIANGLE.—If one of the angles of a triangle is a right angle, the figure is a right triangle. The sides which form the right angle are the LEGS of the triangle, and the third side (opposite the right angle) is the HYPOTENUSE.

The area of a right triangle is always easy to determine. If the base of the triangle is one of its legs, as in figure 17-10 (4), the other leg is the altitude. If the hypotenuse is acting as the base, as in figure 17-10 (3), the triangle can be turned until one of its legs is the base, as in figure 17-10 (1). If the triangle is not known to be a right triangle, then the altitude must be given, as in figure 17-10 (2), in order to calculate the area.

Any triangle whose sides are in the ratio of 3:4:5 is a right triangle. Thus, triangles with sides as follows are right triangles:

Side 1	Side 2	Side 3
3	4	5
6	8	10
12	16	20
3x	4x	5x

(x is any positive number)

In addition to the 3-4-5 triangle, two other types of right triangles occur frequently. Any triangle having one 30° angle and one 60° angle is a right triangle; that is, its third angle is 90°. Any triangle having two 45° angles is a right triangle.

ISOSCELES TRIANGLE.—A triangle having two of its sides equal in length is an ISOSCELES triangle. Since the length of the side opposite an angle is determined by the size of the angle, the isosceles triangle has two equal angles. In figure 17-11 (A), triangle ABC is isosceles. Sides AC and BC are equal in length, and angles A and B are equal.

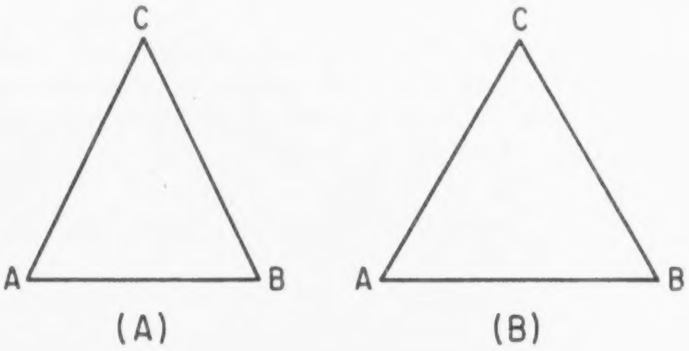


Figure 17-11.—(A) Isoceles triangle; (B) equilateral triangle.

Figure 17-11 (B) illustrates an EQUILATERAL triangle, which is a special case of an isosceles triangle. An equilateral triangle has all three of its sides equal in length. Since the lengths of the sides are directly related to the size of the angles opposite them, an equilateral triangle is also equiangular; that is, all three of its angles are equal.

OBLIQUE TRIANGLES.—Any triangle containing no right angle is an OBLIQUE triangle. Figure 17-12 illustrates two possible configurations, both of which are oblique triangles. An oblique triangle which contains an obtuse angle is often called an OBTUSE triangle.

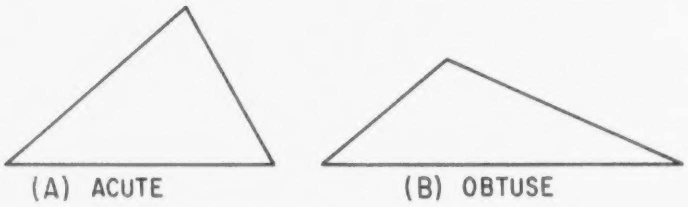


Figure 17-12.—Oblique triangles. (A) Acute; (B) obtuse.

Sum of the Angles

The sum of the angles in any triangle is 180°. For example, if one of the angles is 40° and another is 20°, the third angle is 120°. It is this relationship that justifies the statements made in the preceding section concerning 45° triangles and 30°-60°-90° triangles. If two of the angles are 45° each, then the third angle is 180° - (45°+45°) and the figure is a right triangle. If one angle is 60° and another is 30°, the third angle is 90° and the figure is a right triangle.

QUADRILATERALS

A QUADRILATERAL is a polygon with four sides. The parts of a quadrilateral are its sides, its four angles, and its two DIAGONALS. A diagonal is a straight line joining two alternate vertices of a polygon. Figure 17-13 illustrates the parts of a quadrilateral, in which AC and DB are the diagonals.

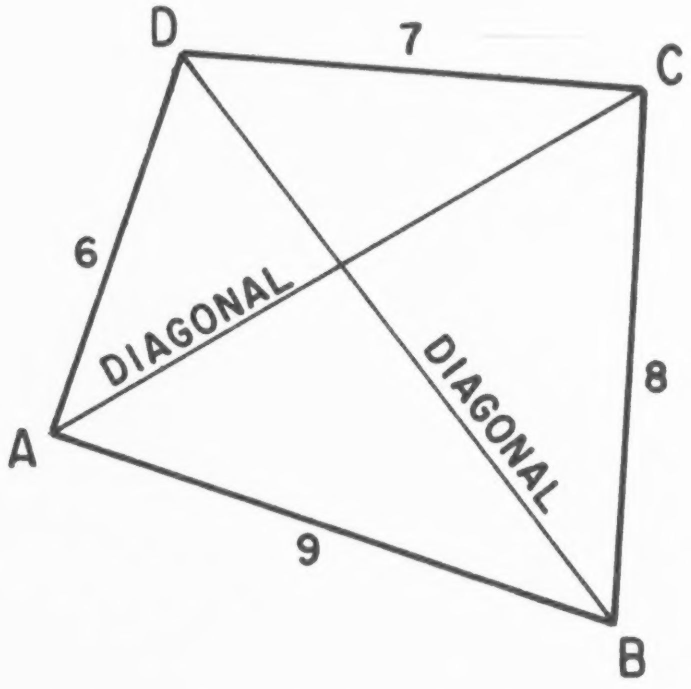


Figure 17-13.—Parts of a quadrilateral.

Perimeter and Area

The perimeter of a quadrilateral is the sum of the lengths of its sides. For example, the perimeter of the quadrilateral in figure 17-13 is 30 units.

The area of a quadrilateral can be found by dividing it into triangles and summing the areas of the triangles. However, the altitudes of the triangles are usually difficult to calculate unless the quadrilateral has at least one pair of parallel sides.

Parallelograms

A PARALLELOGRAM is a quadrilateral in which the opposite sides are parallel. For example, in the parallelogram in figure 17-14, side AB is parallel to side CD. Furthermore, side BC is parallel to side AD.

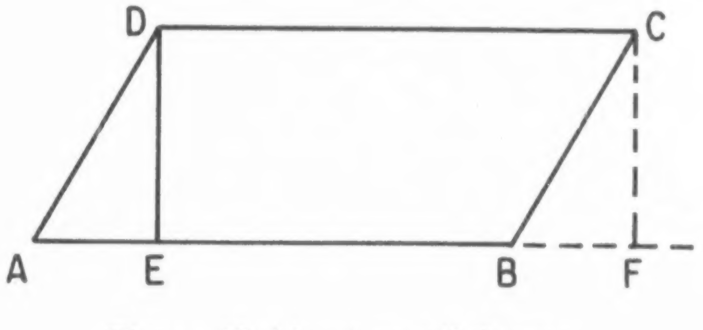


Figure 17-14.—A parallelogram.

Since lines AB and CD are parallel, lines DE and CF (both perpendicular to line AF in figure 17-14) are equal. Angles DAE and CBF in figure 17-14 are equal, because a straight line cutting two parallel lines, such as AD and BC, forms equal angles with the parallel lines. Thus, triangles AED and BFC are equal, and line AD equals line BC. Therefore we have proved that the opposite sides of a parallelogram are equal. If all four of the sides are the same length, the parallelogram is a RHOMBUS.

In addition to the equality of the opposite sides, the opposite angles of a parallelogram are also equal. For example, angle DAB equals angle BCD in figure 17-14, and angle ADC equals angle ABC.

RECTANGLES AND SQUARES.—When all of the angles of a parallelogram are right angles, it is a RECTANGLE. A rectangle with all four of its sides the same length is a SQUARE. Thus a square is a rhombus having 90° angles. Every square is a rectangle, and every rectangle is a parallelogram. Notice that the reverse of this statement is not true.

The area of a rectangle is found by multiplying its length times its width. Therefore, if each side of a square has length s, the area of the square is s^2.

Written as formulas, these areas are as follows:

Rectangle: $A = lw$

or $A = bh$, where b = base,

h = height

Square: $A = s^2$

AREA.—The area of a parallelogram can be found by dividing it into rectangles and triangles. For example, in figure 17-14 the area of the parallelogram is the sum of the areas of triangle AED and figure EBCD. Since triangle AED is equal to triangle BFC, the sum of AED and EBCD is equal to the sum of BFC and EBCD. Thus the area of parallelogram ABCD is the same as the area of rectangle EFCD. Since the area of EFCD is DC multiplied by DE, and DC has the same length as AB, we conclude that the area of a parallelogram is the product of its base times its altitude. Written as a formula, this is

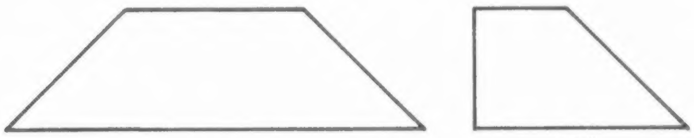
$$A = ba$$

or

$$A = bh, \text{ where h is height}$$

Trapezoids

A TRAPEZOID is a quadrilateral in which two sides are parallel and the other two sides are not parallel. By orienting a trapezoid so that its parallel sides are horizontal, we may call the parallel sides bases. Observe that the bases of a trapezoid are not equal in length. (See fig. 17-15.)



(A) (B)

Figure 17-15.—Typical trapezoids.

The area of a trapezoid may be found by separating it into two triangles and a rectangle, as in figure 17-16. The total area A of the trapezoid is the sum of A_1 plus A_2 plus A_3, and is calculated as follows:

$$\begin{aligned} A &= A_1 + A_2 + A_3 \\ &= \frac{1}{2}ha + hb_1 + \frac{1}{2}hc \\ &= \frac{1}{2}h\ (a + 2b_1 + c) \\ &= \frac{1}{2}h\ (a + b_1 + c) + b_1 \\ &= \frac{1}{2}h\ (b + b_1) \end{aligned}$$

Thus the area of a trapezoid is equal to one-half the altitude times the sum of the bases.

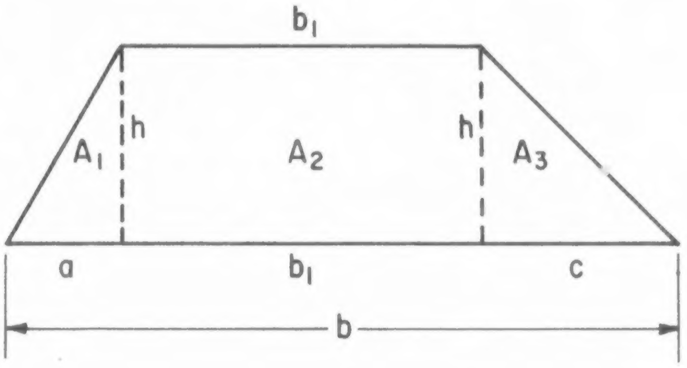


Figure 17-16.—Area of a trapezoid.

Practice problems. Find the area of each of the following figures:

1. Rhombus; base 4 in., altitude 3 in.
2. Rectangle; base 6 ft, altitude 4 ft
3. Parallelogram; base 10 yd, altitude 12 ft
4. Trapezoid; bases 6 ft and 4 ft, altitude 2 yd.

Answers:

1. 12 sq in.
2. 24 sq ft
3. 40 sq yd
4. 30 sq ft

CIRCLES

The mathematical definition of a circle states that it is a plane figure bounded by a curved line, every point of which is equally distant from the center of the figure. The parts of a circle are its circumference, its radius, and its diameter.

Parts of a Circle

The CIRCUMFERENCE of a circle is the line that forms its outer boundary. Circumference is the special term used in referring to the "perimeter" of a circle. (See fig. 17-17.) A RADIUS of a circle is a line joining the center to a point on the circumference, as shown in figure 17-17. A straight line joining two points on the circumference of a circle, and passing through the center, is a DIAMETER. A straight line which touches the circle at just one point is a TANGENT. A tangent is perpendicular to a radius at the point of tangency.

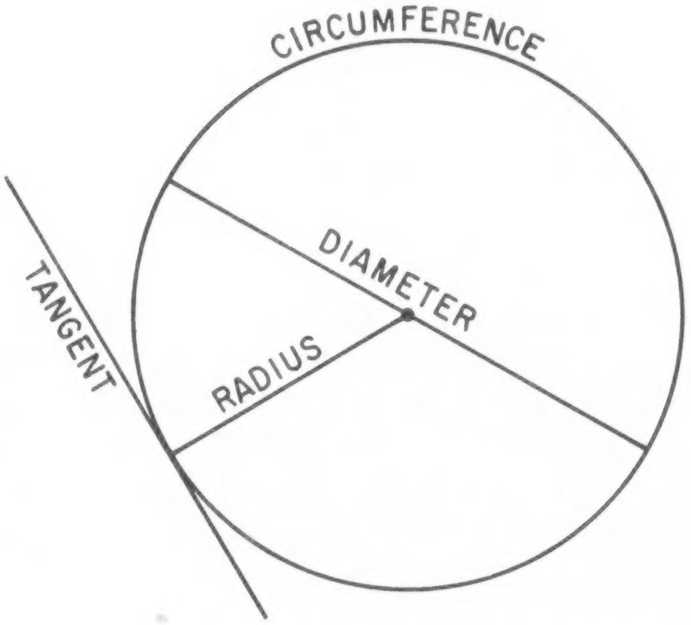


Figure 17-17.—Parts of a circle.

An ARC is a portion of the circumference of a circle. A CHORD is a straight line joining the end points of any arc. The portion of the area of a circle cut off by a chord is a SEGMENT of the circle, and the portion of the circle's area cut off by two radii (radius lines) is a SECTOR. (See fig. 17-18.)

Formulas for Circumference and Area

The formula for the circumference of a circle is based on the relationship between the circumference and the diameter. This comparison can be made experimentally by marking the edge of a circular object, such as a coin, and rolling it (without slippage) along a flat surface. (See fig. 17-19.)

The distance from the initial position to the final position of the disk in figure 17-19 is approximately 3.14 times as long as the diameter of the disk. With any circle, this is always found to be the case; but it is not possible to give the value of C/d (circumference divided by diameter) exactly. The ratio C/d is represented by the symbol π, which is the Greek letter pi. Thus we have the following equations:

$$\frac{C}{d} = \pi$$

$$C = \pi d$$

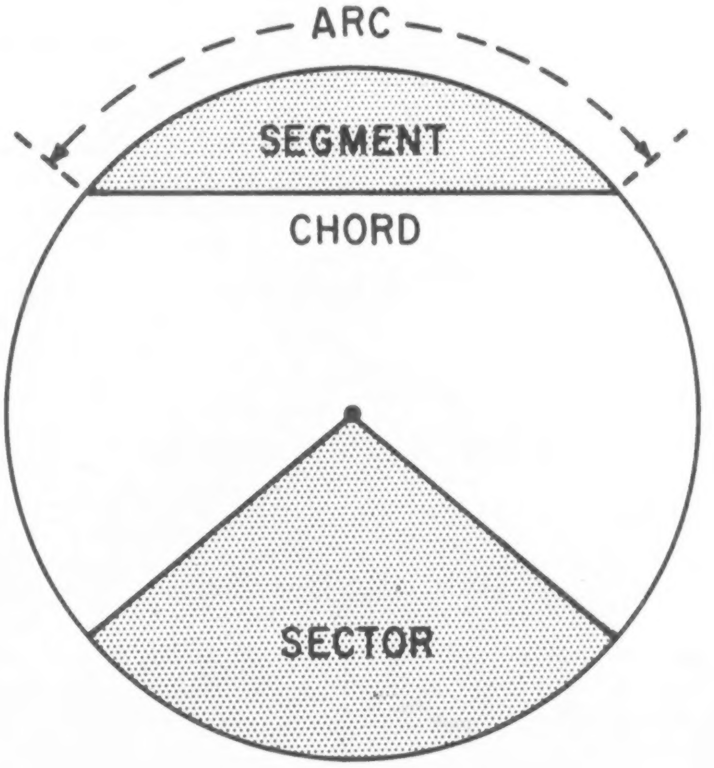


Figure 17-18.—Arc, chord, segment, and sector.

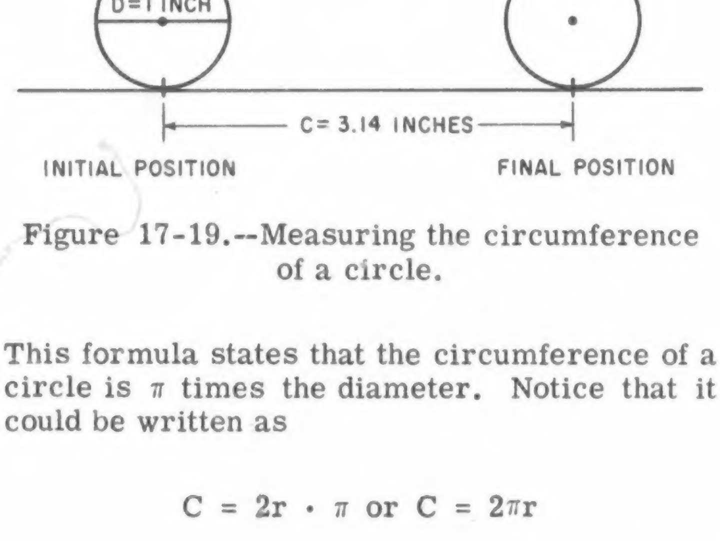


Figure 17-19.--Measuring the circumference of a circle.

This formula states that the circumference of a circle is π times the diameter. Notice that it could be written as

$$C = 2r \cdot \pi \text{ or } C = 2\pi r$$

since the diameter d is the same as 2r (twice the radius).

Although the value of π is not exactly equal to any of the numerical expressions which are sometimes used for it, the ratio is very close to 3.14. If extreme accuracy is required, 3.1416 is used as an approximate value of π. Many calculations involving π are satisfactory if the fraction 22/7 is used as the value of π.

Practice problems. Calculate the circumference of each of the following circles, using 22/7 as the value of π:

1. Radius = 21 in.
2. Diameter = 7.28 in.
3. Radius = 14 ft
4. Diameter = 2.8 yd

Answers:

1. 132 in.
2. 22.88 in.
3. 88 ft
4. 8.8 yd

AREA.—The area of a circle is found by multiplying the square of its radius by π. The formula is written as follows:

$$A = \pi r^2$$

EXAMPLE: Find the area of a circle whose diameter is 4 ft.

SOLUTION: The radius is one-half the diameter. Therefore,

$$\begin{aligned} r &= \frac{1}{2}(4 \text{ ft}) \\ &= 2 \text{ ft} \end{aligned}$$

$$A = \pi r^2 = \pi(2 \text{ ft})^2$$

$$\begin{aligned} A &= 3.14 \ (4 \text{ sq ft}) \\ &= 12.56 \text{ sq ft} \end{aligned}$$

Practice problems. Find the area of each of the following circles:

1. Radius = 7 in.
2. Diameter = 42 mi
3. Diameter = 2.8 ft
4. Radius = 14 yd

Answers:

1. A = 154 sq in.
2. A = 1,386 sq mi
3. 6.16 sq ft
4. 616 sq yd

Concentric Circles

Circles which have a common center are said to be CONCENTRIC. (See fig. 17-20.)

The area of the ring between the concentric circles in figure 17-20 is calculated as follows:

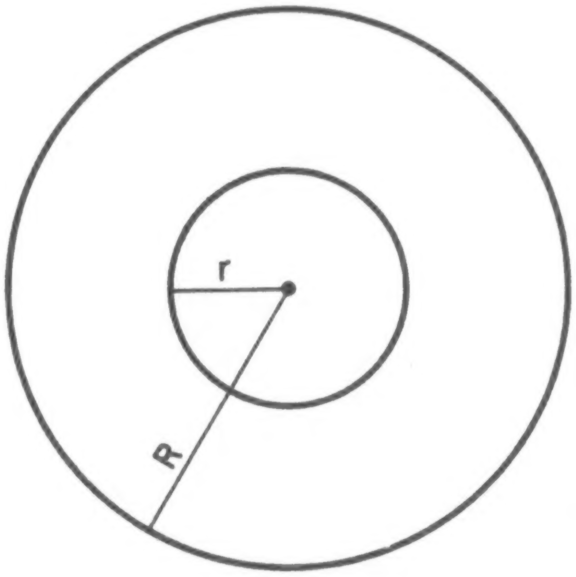


Figure 17-20.—Concentric circles.

Let R = radius of large circle

r = radius of small circle

A_R = area of large circle

A_r = area of small circle

A = area of ring

Then

$$\begin{aligned} A &= A_R - A_r \\ &= \pi R^2 - \pi r^2 \\ &= \pi(R^2 - r^2) \end{aligned}$$

Notice that the last expression is the difference of two squares. Factoring, we have

$$A = \pi(R + r)(R - r)$$

Therefore, the area of a ring between two circles is found by multiplying π times the product of the sum and difference of their radii.

Practice problems. Find the areas of the rings between the following concentric circles:

1. R = 4 in.
   r = 3 in.
2. R = 6 ft
   r = 2 ft

Answers:

1. 22 sq in.
2. 100.6 sq ft

CHAPTER 18

GEOMETRIC CONSTRUCTIONS AND SOLID FIGURES

Many ratings in the Navy involve work which requires the construction or subdivision of geometric figures. For example, materials must be cut into desired shapes, perpendicular lines must be drawn, etc. In addition to these skills, some Navy ratings require the ability to recognize various solid figures and calculate their volumes and surface areas.

CONSTRUCTIONS

From the standpoint of geometry, a CONSTRUCTION may involve either the process of building up a figure or that of breaking down a figure into smaller parts. Some typical constructions are listed as follows:

1. Dividing a line into equal segments.
2. Erecting the perpendicular bisector of a line.
3. Erecting a perpendicular at any point on a line.
4. Bisecting an angle.
5. Constructing an angle.
6. Finding the center of a circle.
7. Constructing an ellipse.

EQUAL DIVISIONS ON A LINE

A line may be divided into any desired number of equal segments by the method shown in figure 18-1.

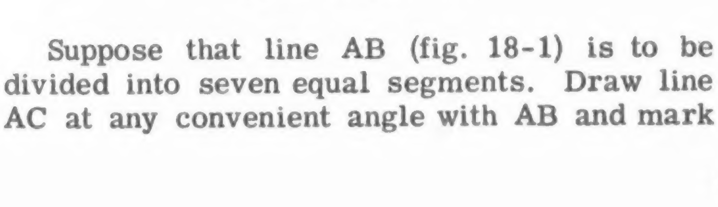


Figure 18-1.—Dividing a line into equal segments.

Suppose that line AB (fig. 18-1) is to be divided into seven equal segments. Draw line AC at any convenient angle with AB and mark off seven spaces of some convenient length, say 1/2 inch, on it. Extend AC, if necessary, in order to get seven intervals of the chosen length on it. This produces the points a, b, c, d, e, f, and g, as shown in figure 18-1. Draw a line from g to B, and then draw lines parallel to gB, starting at each of the points a, b, c, d, e, and f. The segments of AB cut off by these lines are equal in length.

It is frequently necessary to rule a predetermined number of lines on a blank sheet of material. This may be done by a method based on the foregoing discussion. For example, suppose that the sheet of typing paper in figure 18-2 is to be divided into 24 equal spaces.

The 12-inch ruler is laid across the paper at an angle, in such a way that the ends of the

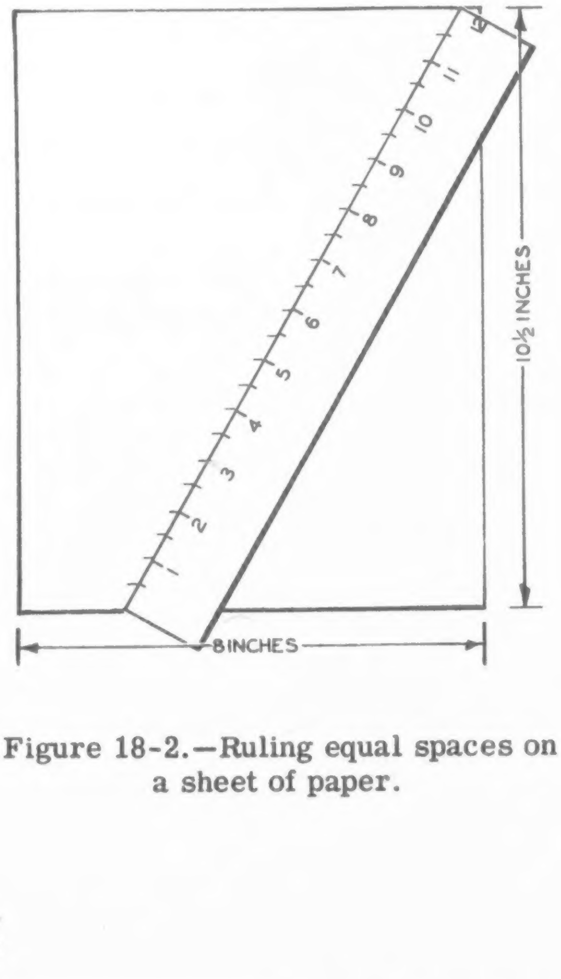


Figure 18-2.—Ruling equal spaces on a sheet of paper.

ruler coincide with the top and bottom edges of the paper. There are 24 spaces, each 1/2 inch wide, on a 12-inch ruler. Therefore, we mark the paper beside each 1/2-inch division marker on the ruler. After removing the ruler, we draw a line through each of the marks on the paper, parallel to the top and bottom edges of the paper.

PERPENDICULAR BISECTOR OF A LINE

To bisect a line or an angle means to divide it into two equal parts. A line may be bisected satisfactorily by measurement, or by a geometric method. If the measuring instrument does not reach the full length of the line, proceed as follows:

1. Starting at one end, measure about half the length of the line and make a mark.
2. Starting at the other end, measure exactly the same distance as before and make a second mark.
3. The bisector of the line lies halfway between these two marks.

The geometric method of bisecting a line is not dependent on measurement. It is based upon the fact that all points equally distant from the ends of a straight line lie on the perpendicular bisector of the line.

Bisecting a line geometrically requires the use of a mathematical compass, which is an instrument for drawing circles and comparing distances. If a line AB is to be bisected as in figure 18-3, the compass is opened until the distance between its points is more than half as long as AB. Then a short arc is drawn above the approximate center of the line and another below, using A as the center of the arcs' circle. (See fig. 18-3.)

Two more short arcs are drawn, one above and one below the approximate center of line AB, this time using B as the center of the arcs' circle.

The two arcs above line AB are extended until they intersect, forming point C, and the two arcs below line AB intersect to form point D. The line joining point C and point D is the perpendicular bisector of line AB.

PERPENDICULAR AT ANY POINT ON A LINE

Figure 18-4 shows a line AB with point C between A and B. A perpendicular to AB is erected at C as follows:

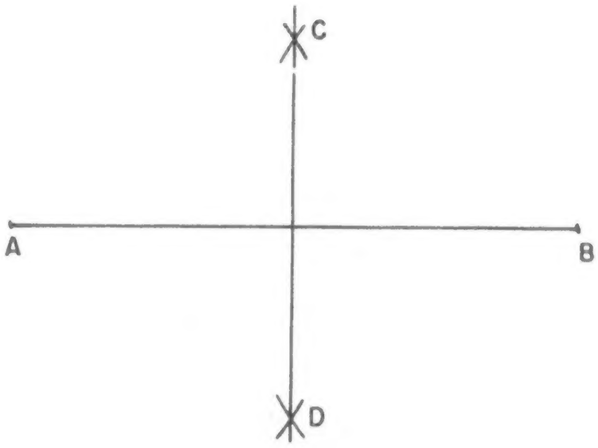


Figure 18-3.—Bisecting a line geometrically.

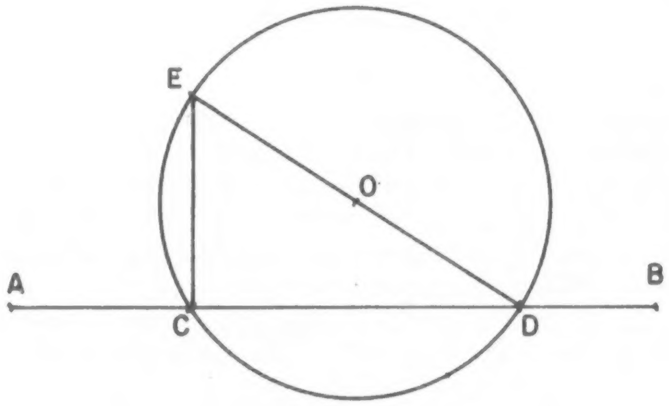


Figure 18-4.—Erecting a perpendicular at a point.

1. Using any convenient point above the line (such as O) as a center, draw a circle with radius OC. This circle cuts AB at C and at D.
2. Draw line DO and extend it to intersect the circle at E.
3. Draw line EC. This line is perpendicular to AB at C.

BISECTING AN ANGLE

Let angle AOB in figure 18-5 be an angle which is to be bisected. Using O as a center and any convenient radius, draw an arc intersecting OA and a second arc intersecting OB. Label these intersections C and D.

Using C and D as centers, and any convenient radius, draw two arcs intersecting halfway between lines OA and OB. A line from O through the intersection of these two arcs is the bisector of angle AOB.

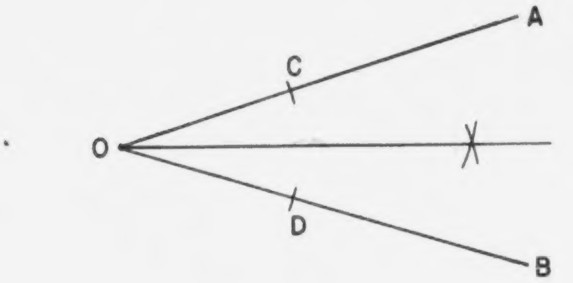


Figure 18-5.—Bisecting an angle.

SPECIAL ANGLES

Several special angles may be constructed by geometric methods, so that an instrument for measuring angles is not necessary in these special cases.

Figure 18-4 illustrates a method of constructing a right angle, DCE, by inscribing a right triangle in a semicircle. But an alternate method is needed for those situations in which drawing circles is inconvenient. The method described herein makes use of a right triangle having its sides in the ratio of 3 to 4 to 5. It is often used in laying out the foundations of buildings. The procedure is as follows:

1. A string is stretched as shown in figure 18-6, forming line AC. The length of AC is 3 feet.
2. A second string is stretched, crossing line AC at A, directly above the point intended as the corner of the foundation. Point D on this line is 4 feet from A.
3. Attach a third string, 5 feet long, at C and D. When AC and AD are spread so that line CD is taut, angle DAC is a right angle.

A 60° angle is constructed as shown in figure 18-7. With AB as a radius and A and B as centers, draw arcs intersecting at C. When A and B are connected to C by straight lines, all three angles of triangle ABC are 60° angles.

The special angles already discussed are used in constructing 45° and 30° angles. A 90° angle is bisected to form two 45° angles, and a 60° angle is bisected to form two 30° angles.

FINDING THE CENTER OF A CIRCLE

It is sometimes necessary to find the center of a circle of which only an arc or a segment is given. (See fig. 18-8.)

From any point on the arc, such as A, draw two chords intersecting the arc in any two points, such as B and C. With the points A, B, and C as centers, use any convenient radius and draw short intersecting arcs to form the perpendicular bisectors of chords AC and AB. Join the intersecting arcs on each side of AC, obtaining line MP, and join the arcs on each side of AB, obtaining line NQ. The intersection of MP and NQ is point O, the center of the circle.

ELLIPSES

An ellipse of specified length and width is constructed as follows:

1. Draw the major axis, AB, and the minor axis, CD, as shown in figure 18-9.
2. On a straightedge or ruler, mark a point (labeled a in the figure) and from this point measure one-half the length of the minor axis and make a second mark (b in figure 18-9). From point a, measure one-half the length of

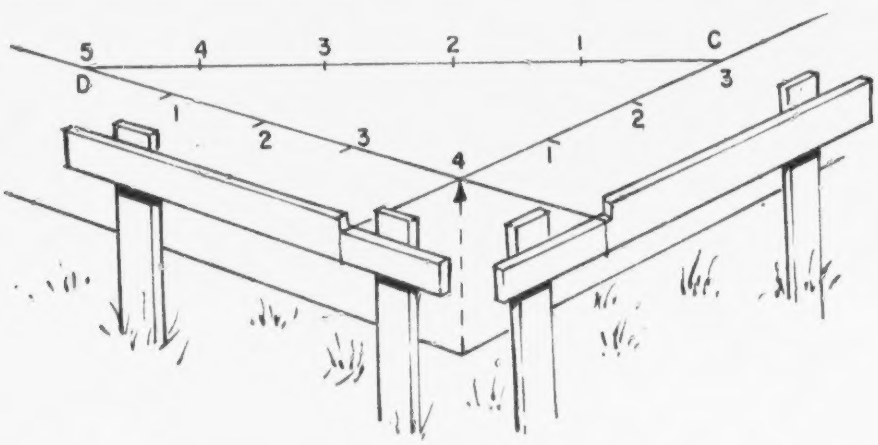


Figure 18-6.—Constructing a right angle by the 3-4-5 method.

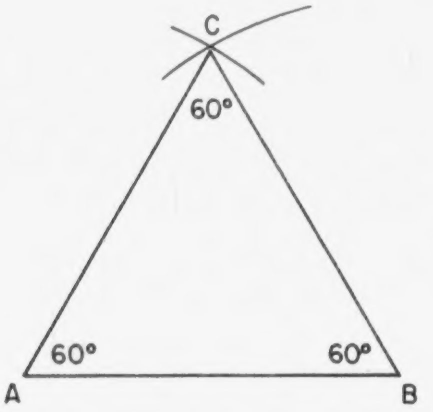


Figure 18-7.—Constructing 60° angles.

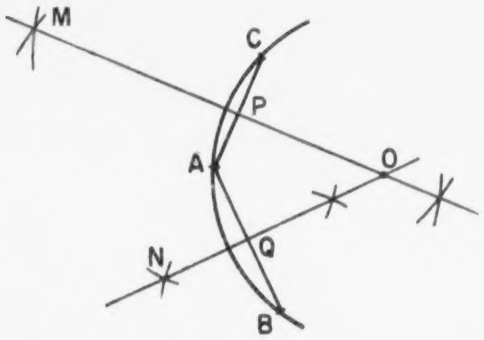


Figure 18-8.—Finding the center of a circle.

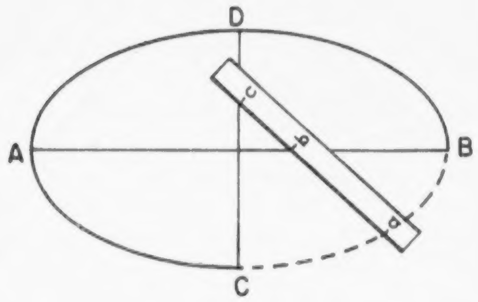


Figure 18-9.—Constructing an ellipse.

the major axis and make a third mark (c in the figure).

3. Place the straightedge on the axes so that b lies on the major axis and c lies on the minor axis. Mark the paper with a dot beside point a. Reposition the straightedge, keeping b on the major axis and c on the minor axis, and make a dot beside the new position of a.

4. After locating enough dots to see the elliptical pattern, join the dots with a smooth curve.

SOLID FIGURES

The plane figures discussed in chapter 17 of this course are combined to form solid figures. For example, three rectangles and two triangles may be combined as shown in figure 18-10. The flat surfaces of the solid figure are its FACES; the top and bottom faces are the BASES, and the faces forming the sides are the LATERAL FACES.

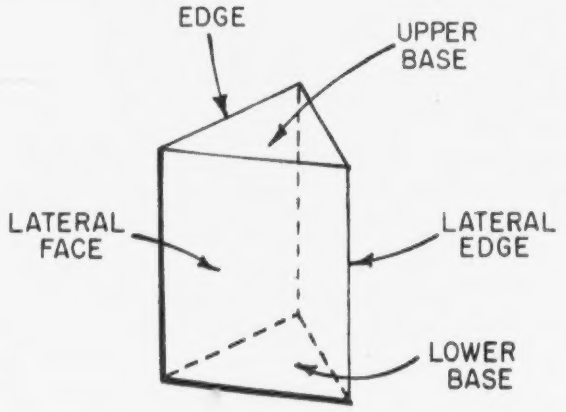


Figure 18-10.—Parts of a solid figure.

Some solid figures do not have any flat faces, and some have a combination of curved surfaces and flat surfaces. Examples of solids with curved surfaces include cylinders, cones, and spheres.

PRISMS

The solid shown in figure 18-10 is a PRISM. A prism is a solid with three or more lateral faces which intersect in parallel lines.

Types of Prisms

The name of a prism depends upon its base polygons. If the bases are triangles, as in figure 18-10, the figure is a TRIANGULAR prism. A RECTANGULAR prism has bases which are rectangles.

If the bases of a prism are perpendicular to the planes forming its lateral faces, the prism is a RIGHT prism.

A PARALLELEPIPED is a prism with parallelograms for bases. Since the bases are parallel to each other, this means that they cut the lateral faces to form parallelograms. Therefore, in a parallelepiped, all of the faces are parallelograms. If a parallelepiped is a right prism, and if its bases are rectangles, it is a rectangular solid. A CUBE is a rectangular solid in which all of the six rectangular faces are squares.

Parts of a Prism

The parts of a prism are shown in figure 18-10. The line formed by the joining of two faces of a prism is an EDGE. If the two faces forming an edge are lateral faces, the edge thus formed is a LATERAL EDGE.

Surface Area and Volume

The SURFACE AREA of a prism is the sum of the areas of all of its faces, including the bases. The VOLUME of a prism may be considered as the sum of the volumes of many thin wafers, each having a thickness of one unit and a shape that duplicates the shape of the base. (See fig. 18-11.)

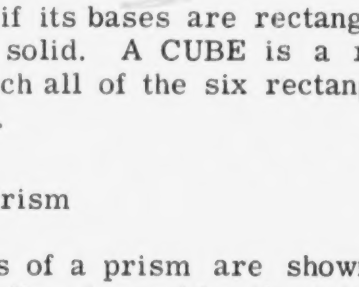


Figure 18-11.—Volume of a prism.

The wafers which comprise the prism in figure 18-11 all have the same area, which is the area of the base. Therefore, the volume of the prism is found by multiplying the area of the base times the number of wafers. Since each wafer is 1 inch thick, the number of wafers is the same as the height of the prism in inches. The resulting formula for the volume of a prism, using B to represent the area of the base and h to represent the height, is as follows:

$$V = Bh$$

When a prism has lateral edges which are not perpendicular to the bases, the height of the prism is the perpendicular distance between the bases. (See fig. 18-12.) The formula for the volume remains the same, even though the prism is no longer a right prism.

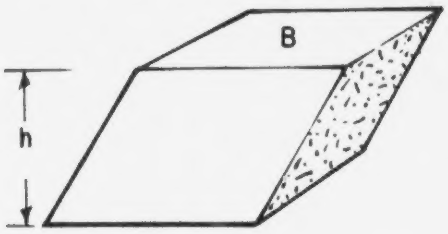


Figure 18-12.—Height of a prism which is not a right prism.

CIRCULAR CYLINDERS

Any surface may be considered as the result of moving a straight line in a direction at right angles to its length. For example, suppose that the stick of charcoal in figure 18-13 is moved from position AB to position CD by dragging it across the paper. The broad mark made by the charcoal represents a plane surface. The surface is said to be "generated" by moving line AB.

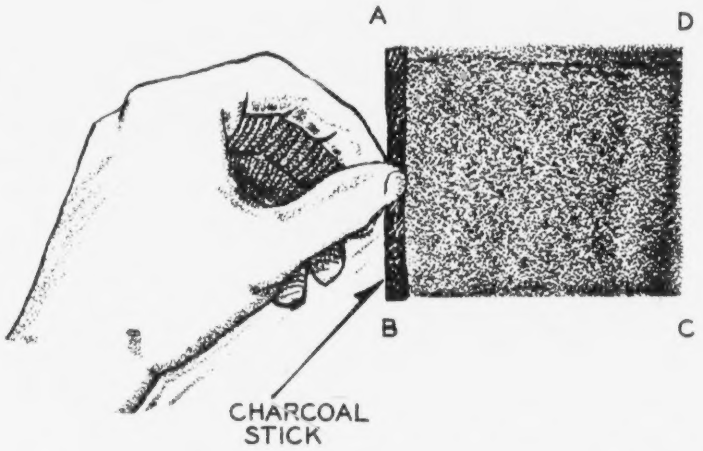


Figure 18-13.—Surface generated by a moving line.

The movement of the line in figure 18-13 may be controlled by requiring that its lower end trace a particular path. For example, if line AB moves so as to trace an ellipse as in figure 18-14 (A), a cylindrical surface is generated by the line. This surface, shown in figure 18-14 (B), is an elliptical cylinder.

Any line in the surface, parallel to the generating line, such as CD or EF in figure 18-14 (B), is an ELEMENT of the cylinder. If the elements are perpendicular to the bases, the cylinder is a RIGHT CYLINDER. If the

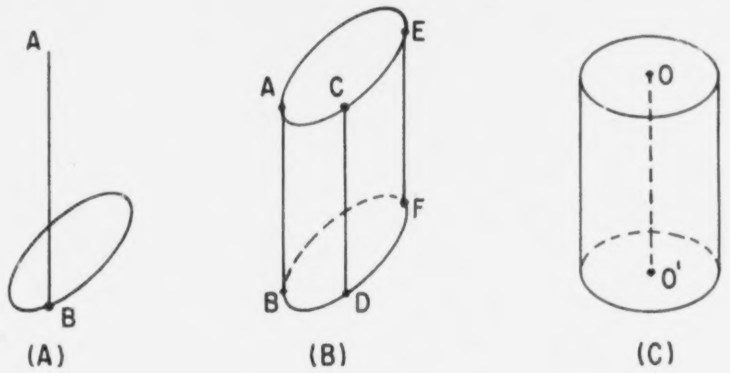


Figure 18-14.—(A) Line generating a cylinder; (B) elliptical cylinder; (C) circular cylinder.

bases are circles, the cylinder is a CIRCULAR CYLINDER. Figure 18-14 (C) illustrates a right circular cylinder. Line O-O', joining the centers of the bases of a right circular cylinder, is the AXIS of the cylinder.

Surface Area and Volume

The lateral area of a cylinder is the area of its curved surface, excluding the area of its bases. Figure 18-15 illustrates an experimental method of determining the lateral area of a right circular cylinder.

The card of length L and width W in figure 18-15 is rolled into a cylinder. The height of the cylinder is W and the circumference is L. The lateral area is the same as the original area of the card, LW. Therefore, the lateral area of the cylinder is found by multiplying its height by the circumference of its base. Written as a formula, this is

$$A = Ch$$

EXAMPLE: Find the lateral area of a right circular cylinder whose base has a radius of 4 inches and whose height is 6 inches.

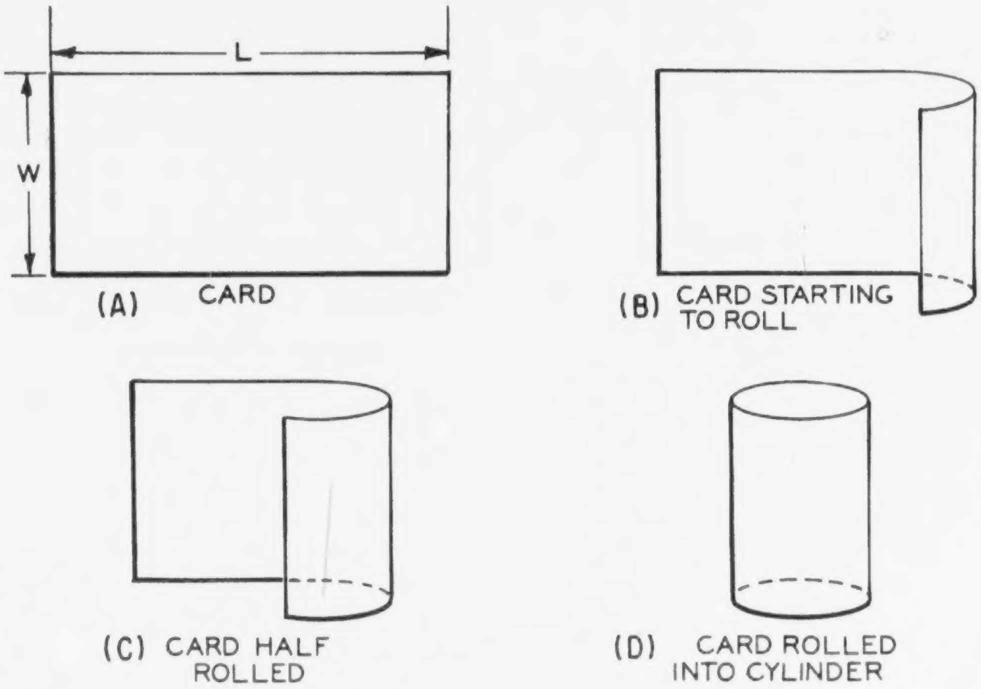


Figure 18-15.—Lateral area of a cylinder.

SOLUTION: The circumference of the base is

$$C = \pi d$$

$$C = 3.14 \times 8 \text{ in.}$$

$$= 25.12 \text{ in.}$$

Therefore,

$$A = 25.12 \text{ in.} \times 6 \text{ in.}$$

$$= 151 \text{ sq in. (approximately)}$$

The formula for the volume of a cylinder is obtained by the same reasoning process that was used for prisms. The cylinder is considered to be composed of many circular wafers, or disks, each one unit thick. The area of each disk, multiplied by the number of disks, is the volume of the cylinder. With V representing volume, A representing the area of each disk, and n representing the number of disks, the formula is as follows:

$$V = An$$

Since the number of disks is the same as the height of the cylinder, the formula for the volume of a cylinder is normally written

$$V = Bh$$

In this formula, B is the area of the base and h is the height of the cylinder.

EXAMPLE: Determine the volume of a circular cylinder with a base of radius 5 inches and a height of 14 inches.

SOLUTION:

$$\begin{aligned} V &= Bh \\ &= (\pi \times 5^2) \times 14 \\ &= \frac{22}{7} \times 25 \times 14 \\ &= 22 \times 25 \times 2 \\ &= 22 \times 50 \\ &= 1{,}100 \text{ cu in.} \end{aligned}$$

Practice problems:

1. Determine the lateral area of a right circular cylinder with a base of diameter 7 inches and a height of 4 inches.

2. Determine the volume of the cylinder in problem 1.

Answers:

1. 88 sq in. 2. 154 cu in.

REGULAR PYRAMIDS AND RIGHT CIRCULAR CONES

A PYRAMID is a solid figure, the lateral faces of which are triangles. (See fig. 18-16.) A REGULAR PYRAMID has all of its lateral faces equal.

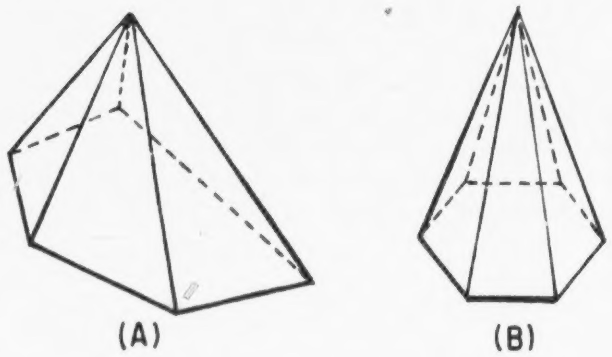


Figure 18-16.—(A) Irregular pyramid; (B) regular pyramid.

A regular pyramid with a very large number of lateral faces would have a base polygon with many sides. If the number of sides is sufficiently large, the base polygon is indistinguishable from a circle and the surface formed by the many lateral faces becomes a smoothly curved surface. The solid figure thus formed is a RIGHT CIRCULAR CONE. (See fig. 18-17.)

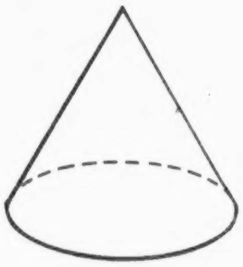


Figure 18-17.—Right circular cone.

Slant Height

The slant height of a regular pyramid is the perpendicular distance from the vertex to the center of any side of the base. For example, the length of line AV in figure 18-18 (A) is the slant height. The slant height of a right circular cone is the length of any straight line joining the vertex to the circumference line of the base. Such a line is perpendicular to a line tangent to the base at the point where the slant height intersects the base. (See fig. 18-18 (B).) Lines AV, BV, and CV in figure 18-18 (B) are all slant heights.

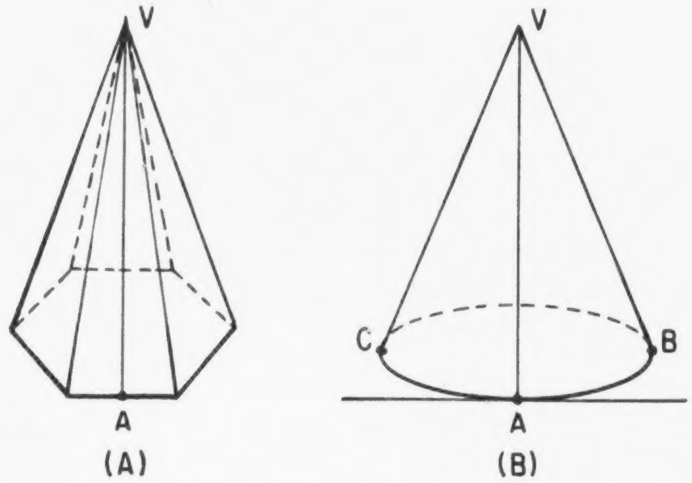


Figure 18-18.—(A) Slant height of a regular pyramid; (B) slant height of a right circular cone.

Lateral Area

The lateral area of a pyramid is the sum of the areas of its lateral faces. If the pyramid is regular, its lateral faces have equal bases; furthermore, the slant height is the altitude of each face. Therefore, the area of each lateral face is one-half the slant height multiplied by the length of one side of the base polygon. Since the sum of these sides is the perimeter of the base, the total lateral area of the pyramid is the product of one-half its slant height multiplied by the perimeter of its base. Using s to represent slant height and P to represent the perimeter of the base, the formula is as follows:

$$\text{Lateral Area} = \frac{1}{2}\,sP$$

A right circular cone can be considered as a regular pyramid with an infinite number of faces. Therefore, using C to represent the circumference of the base, the formula for the lateral area of a right circular cone is

$$\text{Lateral Area} = \frac{1}{2}\,sC$$

Volume

The volume of a pyramid is determined by its base and its altitude, as is the case with other solid figures. Experiments show that the volume of any pyramid is one-third of the product of its base and its altitude. This may be stated as a formula with V representing volume, B representing the area of the base, and h representing height (altitude), as follows:

$$V = \frac{1}{3}\,Bh$$

The formula for the volume of a pyramid does not depend in any way upon the number of faces. Therefore, we use the same formula for the volume of a right circular cone. Since the base is a circle, we replace B with πr^2 (where r is the radius of the base). The formula for the volume of a right circular cone is then

$$\begin{aligned} V &= \frac{1}{3}\,Bh \\ &= \frac{1}{3}\,\pi r^2 h \end{aligned}$$

Practice problems:

1. Find the lateral area of a regular pyramid with a 5-sided base measuring 3 inches on each side, if the slant height is 12 inches.

2. Find the lateral area of a right circular cone whose base has a diameter of 6 cm and whose slant height is 14 cm.

3. Find the volume of a regular pyramid with a square base measuring 4 cm on each side, if the vertex is 9 cm above the base.

4. Find the volume of a right circular cone whose base has a diameter of 14 inches, if the altitude is 21 inches.

Answers:

1. 90 sq in.
2. 132 sq cm
3. 48 cu cm
4. 1078 cu in.

SPHERES

A SPHERE is a solid figure with all points on its surface equally distant from its center.

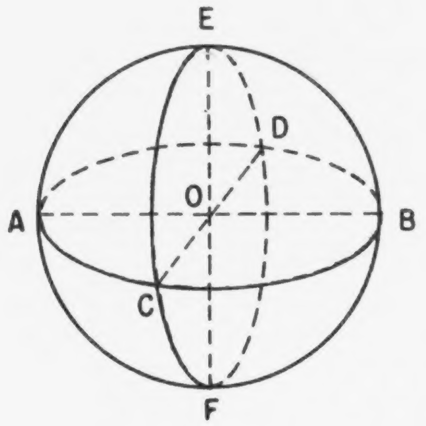


Figure 18-19.—Parts of a sphere.

In figure 18-19, the center of the sphere is point O.

A RADIUS of a sphere is a straight line segment joining the center of the sphere to a point on the surface. Lines OA, OB, OC, OD, OE, and OF in figure 18-19 are radii. A DIAMETER of a sphere is a straight line segment joining two points on the surface and passing through the center of the sphere. Lines AB, CD, and EF in figure 18-19 are diameters. A HEMISPHERE is half of a sphere.

Circles of various sizes may be drawn on the surface of a sphere. The largest circle that may be so drawn is one with a radius equal to the radius of the sphere. Such a circle is a GREAT CIRCLE. In figure 18-19, circles AEBF, ACBD, and CEDF are great circles.

On the surface of a sphere, the shortest distance between two points is an arc of a great circle drawn so that it passes through the two points. This explains the importance of great circles in the science of navigation, since the earth is approximately a sphere.

Surface Area

The surface area of a sphere may be calculated by multiplying 4 times π times the square of the radius. Written as a formula, this is

$$A = 4\pi r^2$$

The formula for the surface area of a sphere may be rewritten as follows:

$$A = (2\pi r)(2r)$$

When the formula is factored in this way, it is easy to see that the surface area of a sphere is simply its circumference times its diameter.

Volume

The volume of a sphere whose radius is r is given by the formula

$$V = \frac{4}{3}\pi r^3$$

EXAMPLE: Find the volume of a sphere whose diameter is 42 inches.

SOLUTION:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3} \times \frac{22}{7} \times (21 \text{ in.})^3 \\ &= \frac{4}{3} \times \frac{22}{7} \times 21 \times 21 \times 21 \text{ cu in.} \\ &= 88 \times 21 \times 21 \text{ cu in.} \\ &= 38{,}808 \text{ cu in.} \end{aligned}$$

Practice problems. Calculate the surface area and the volume of the sphere in each of the following problems:

1. Radius = 7 inches
2. Radius = 14 cm

Answers:

1. Area = 616 sq in.
   Volume = 1,437 cu in. (approx.)
2. Area = 2,464 sq cm
   Volume = 11,499 cu cm (approx.)

CHAPTER 19

NUMERICAL TRIGONOMETRY

The word "trigonometry" means "measurement by triangles." As it is presented in many textbooks, trigonometry includes topics other than triangles and measurement. However, this chapter is intended only as an introduction to the numerical aspects of trigonometry as they relate to measurement of lengths and angles.

SPECIAL PROPERTIES OF RIGHT TRIANGLES

A RIGHT TRIANGLE has been defined as any triangle containing a right angle. The side opposite the right angle in a right triangle is a HYPOTENUSE. (See fig. 19-1.) In figure 19-1, side AC is the hypotenuse.

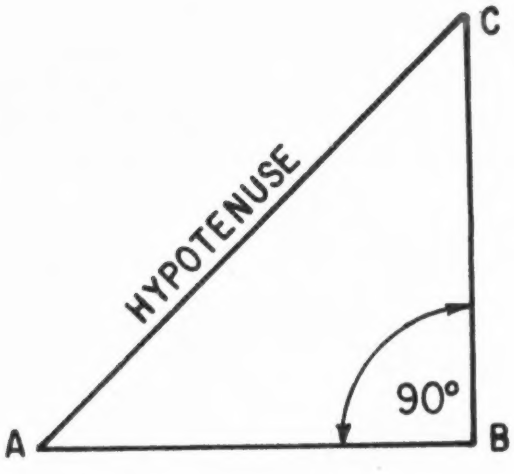


Figure 19-1.—A right triangle.

An important property of all right triangles, which relates the lengths of the three sides, was discovered by the Greek philosopher Pythagoras.

PYTHAGOREAN THEOREM

The rule of Pythagoras, or PYTHAGOREAN THEOREM, states that the square of the length of the hypotenuse (in any right triangle) is equal to the sum of the squares of the lengths of the other two sides. For example, if the sides are labeled as in figure 19-2 (A), the Pythagorean Theorem is stated in symbols as follows:

$$x^2 + y^2 = r^2$$

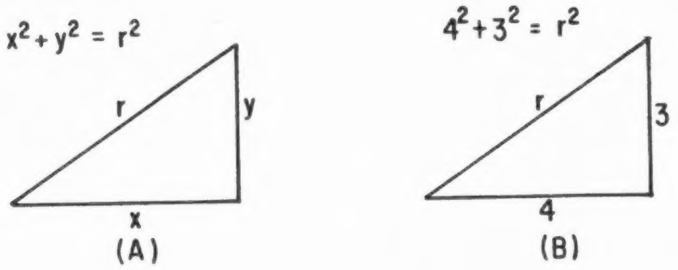


Figure 19-2.—The Pythagorean Theorem. (A) General triangle; (B) triangle with sides of specific lengths.

An example of the use of the Pythagorean Theorem in a problem follows:

EXAMPLE: Find the length of the hypotenuse in the triangle shown in figure 19-2 (B).

SOLUTION:

$$r^2 = 3^2 + 4^2$$
$$r = \sqrt{9 + 16}$$
$$= \sqrt{25} = 5$$

EXAMPLE: An observer on a ship at point A, figure 19-3, knows that his distance from point C is 1,200 yards and that the length of BC is 1,300 yards. He measures angle A and finds that it is 90°. Calculate the distance from A to B.

SOLUTION: By the rule of Pythagoras,

$$(BC)^2 = (AB)^2 + (AC)^2$$
$$(1{,}300)^2 = (AB)^2 + (1{,}200)^2$$
$$(1{,}300)^2 - (1{,}200)^2 = (AB)^2$$
$$(13 \times 10^2)^2 - (12 \times 10^2)^2 = (AB)^2$$
$$(169 \times 10^4) - (144 \times 10^4) = (AB)^2$$
$$25 \times 10^4 = (AB)^2$$
$$500 \text{ yd} = AB$$

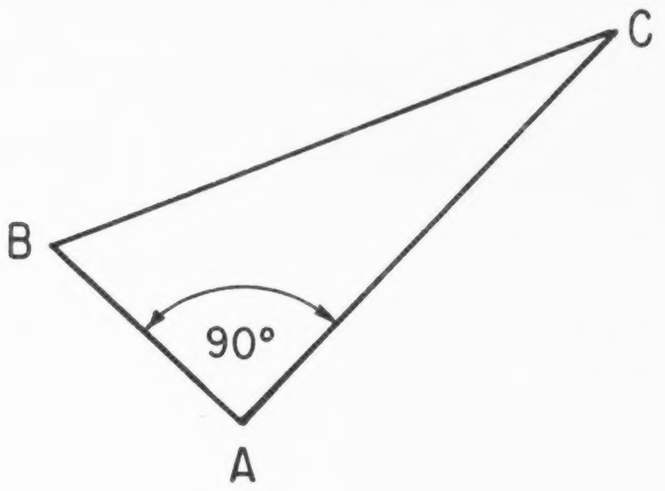


Figure 19-3.—Using the Pythagorean Theorem.

SIMILAR RIGHT TRIANGLES

Two right triangles are SIMILAR if one of the acute angles of the first is equal to one of the acute angles of the second. This conclusion is supported by the following reasons:

1. The right angle in the first triangle is equal to the right angle in the second, since all right angles are equal.
2. The sum of the angles of any triangle is 180°. Therefore, the sum of the two acute angles in a right triangle is 90°.
3. Let the equal acute angles in the two triangles be represented by A and A' respectively. (See fig. 19-4.) Then the other acute angles, B and B', are as follows:

$$B = 90° - A$$
$$B' = 90° - A'$$

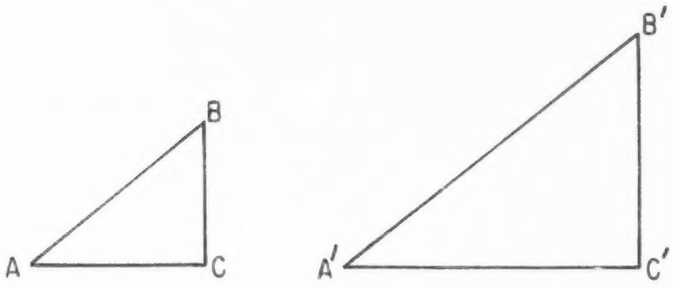


Figure 19-4.—Similar right triangles.

4. Since angles A and A' are equal, angles B and B' are also equal.
5. We conclude that two right triangles with one acute angle of the first equal to one acute angle of the second have all of their corresponding angles equal. Thus the two triangles are similar.

Practical situations frequently occur in which similar right triangles are used to solve problems. For example, the height of a tree can be determined by comparing the length of its shadow with that of a nearby flagpole, as shown in figure 19-5.

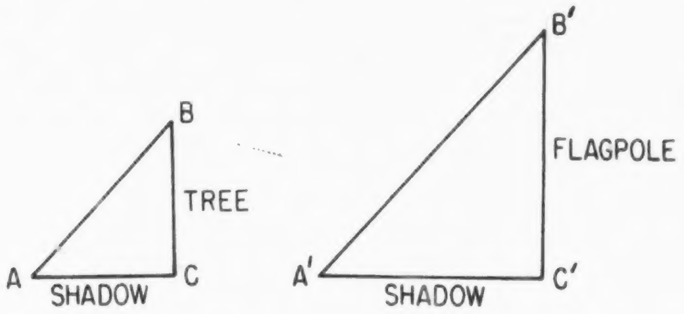


Figure 19-5.—Calculation of height by comparison of shadows.

Assume that the rays of the sun are parallel and that the tree and flagpole both form 90° angles with the ground. Then triangles ABC and A'B'C' are right triangles and angle B is equal to angle B'. Therefore, the triangles are similar and their corresponding sides are proportional, with the following result:

$$\frac{BC}{AC} = \frac{B'C'}{A'C'}$$

$$BC = \frac{(AC) \times (B'C')}{A'C'}$$

Suppose that the flagpole is known to be 30 feet high, the shadow of the tree is 12 feet long, and the shadow of the flagpole is 24 feet long. Then

$$BC = \frac{12 \times 30}{24} = 15 \text{ feet}$$

Practice problems.

1. A mast at the top of a building casts a shadow whose tip is 48 feet from the base of the building. If the building is 12 feet high and its shadow is 32 feet long, what is the length of the mast? (NOTE: If the length of the mast is x, then the height of the mast above the ground is x + 12.)

2. Figure 19-6 represents an L-shaped building with dimensions as shown. On the line of sight from A to D, a stake is driven at C, a point 8 feet from the building and 10 feet from A. If ABC is a right angle, find the length of AB and the length of AD. Notice that AE is 18 feet and ED is 24 feet.

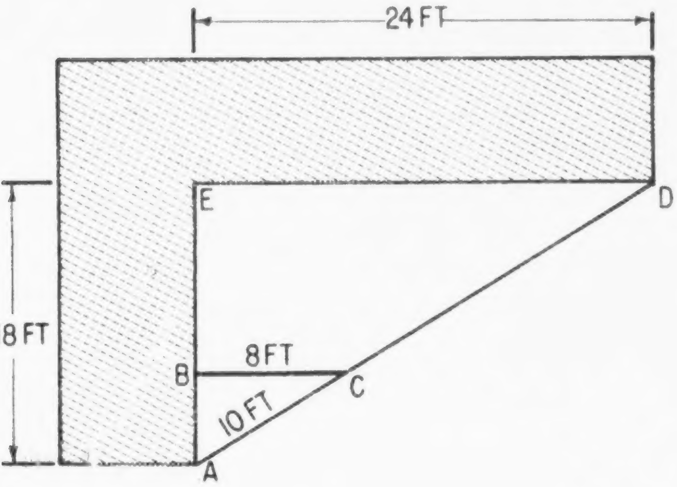


Figure 19-6.—Using similar triangles.

Answers:

1. 6 feet

2. AB = 6 feet
AD = 30 feet

TRIGONOMETRIC RATIOS

The relationships between the angles and the sides of a right triangle are expressed in terms of TRIGONOMETRIC RATIOS. For example, in figure 19-7, the sides of the triangle are named in accordance with their relationship to angle θ. In trigonometry, angles are usually named by means of Greek letters. The Greek name of the symbol θ is theta.

The six trigonometric ratios for the angle θ are listed in table 19-1.

The ratios are defined as follows:

1. $\sin\theta = \dfrac{\text{side opposite } \theta}{\text{hypotenuse}} = \dfrac{y}{r}$

2. $\cos\theta = \dfrac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \dfrac{x}{r}$

3. $\tan\theta = \dfrac{\text{side opposite } \theta}{\text{side adjacent to } \theta} = \dfrac{y}{x}$

4. $\cot\theta = \dfrac{\text{side adjacent to } \theta}{\text{side opposite } \theta} = \dfrac{x}{y}$

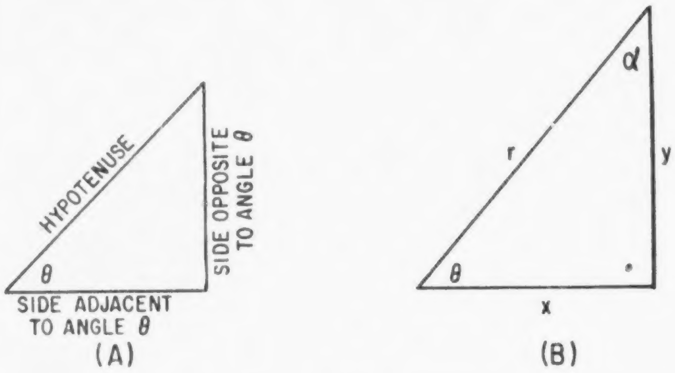


Figure 19-7.—Relationship of sides and angles in a right triangle. (A) Names of the sides; (B) symbols used to designate the sides.

Table 19-1.—Trigonometric ratios.

Name of ratio	Abbreviation
sine of θ	$\sin\theta$
cosine of θ	$\cos\theta$
tangent of θ	$\tan\theta$
cotangent of θ	$\cot\theta$
secant of θ	$\sec\theta$
cosecant of θ	$\csc\theta$

5. $\sec\theta = \dfrac{\text{hypotenuse}}{\text{side adjacent to } \theta} = \dfrac{r}{x}$

6. $\csc\theta = \dfrac{\text{hypotenuse}}{\text{side opposite to } \theta} = \dfrac{r}{y}$

The other acute angle in figure 19-7 (B) is labeled α (Greek alpha). The side opposite α is x and the side adjacent to α is y. Therefore the six ratios for α are as follows:

1. $\sin\alpha = \dfrac{x}{r}$

2. $\cos\alpha = \dfrac{y}{r}$

3. $\tan\alpha = \dfrac{x}{y}$

4. $\cot\alpha = \dfrac{y}{x}$

5. $\sec\alpha = \dfrac{r}{y}$

6. $\csc\alpha = \dfrac{r}{x}$

Suppose that the sides of triangle (B) in figure 19-7 are as follows: $x = 3$, $y = 4$, $r = 5$. Then each of the ratios for angles θ and α may be expressed as a common fraction or as a decimal. For example,

$$\sin \theta = \frac{4}{5} = 0.800$$

$$\sin \alpha = \frac{3}{5} = 0.600$$

Decimal values have been computed for ratios of angles between 0° and 90°, and values for angles above 90° can be expressed in terms of these same values by means of conversion formulas. Appendix II of this training course gives the sine, cosine, and tangent of angles from 0° to 90°. The secant, cosecant, and cotangent are calculated, when needed, by using their relationships to the three principal ratios. These relationships are as follows:

$$\text{secant } \theta = \frac{1}{\text{cosine } \theta}$$

$$\text{cosecant } \theta = \frac{1}{\text{sine } \theta}$$

$$\text{cotangent } \theta = \frac{1}{\text{tangent } \theta}$$

TABLES

Tables of decimal values for the trigonometric ratios may be constructed in a variety of ways. Some give the angles in degrees, minutes, and seconds; others in degrees and tenths of a degree. The latter method is more compact and is the method used for appendix II. The "headings" at the bottom of each page in appendix II provide a convenient reference showing the minute equivelents for the decimal fractions of a degree. For example, 12' (12 minutes) is the equivalent of 0.2°.

Finding the Function Value

The trigonometric ratios are sometimes called **FUNCTIONS**, because the value of the ratio depends upon (is a function of) the angle size. Finding the function value in appendix II is easily accomplished. For example, the sine 35° is found by looking in the "sin" row opposite the large number 35, which is located in the extreme left-hand column.

Since our angle in this example is exactly 35°, we look for the decimal value of the sine in the column with the 0.0° heading. This column contains decimal values for functions of the angle plus 0.0°; in our example, 35° plus 0.0°, or simply 34.0°. Thus we find that the sine of 35.0° is 0.5736. By the same reasoning, the sine of 42.7° is 0.6782, and the tangent of 32.3° is 0.6322.

A typical problem in trigonometry is to find the value of an unknown side in a right triangle when only one side and one acute angle are known. EXAMPLE: In triangle ABC (fig. 19-8), find the length of AC if AB is 13 units long and angle CAB is 34.7°.

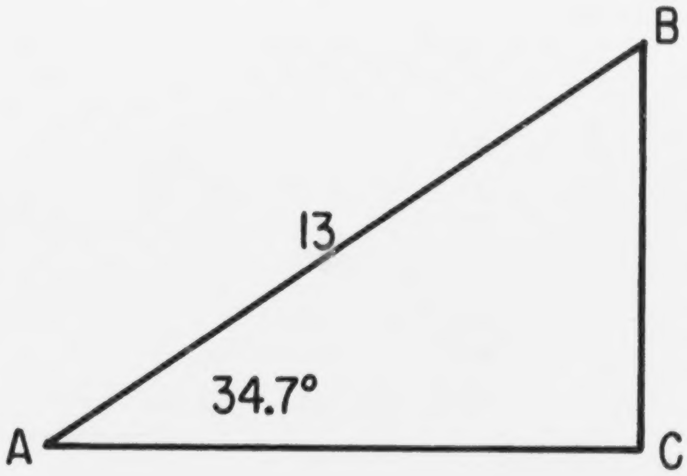


Figure 19-8.—Using the trigonometric ratios to evaluate the sides.

SOLUTION:

$$\frac{AC}{13} = \cos 34.7°$$

$$\begin{aligned} AC &= 13 \cos 34.7° \\ &= 13 \times 0.8221 \\ &= 10.69 \text{ (approx.)} \end{aligned}$$

The angles of a triangle are frequently stated in degrees and minutes, rather than degrees and tenths. For example, in the foregoing problem, the angle might have been stated as 34°42'. When the stated number of minutes is an exact multiple of 6 minutes, the minute entries at the bottom of each page in appendix II may be used.

Finding the Angle

Problems are frequently encountered in which two sides are known, in a right triangle, but neither of the actue angles is known. For example, by applying the Pythagorean Theorem we can verify that the triangle in figure 19-9 is

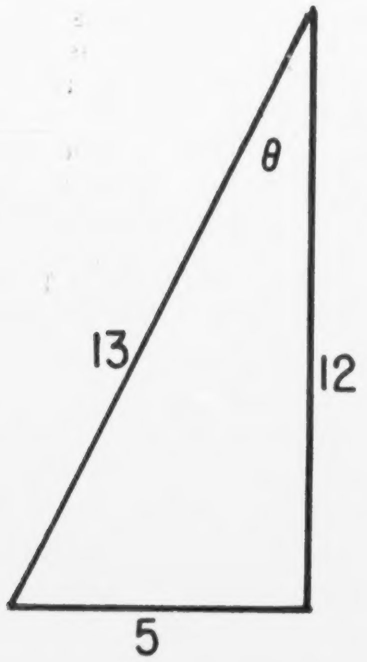


Figure 19-9.—Using trigonometric ratios to evaluate angles.

a right triangle. The only information given, concerning angle θ, is the ratio of sides in the triangle. The size of θ is calculated as follows:

$$\tan \theta = \frac{5}{12} = 0.4167$$

$$\theta = \text{the angle whose tangent is } 0.4167$$

Assuming that the sides and angles in figure 19-9 are in approximately the correct proportions, we estimate that angle θ is about 20°. The table entries for the tangent in the vicinity of 20° are slightly too small, since we need a number near 0.4167. However, the tangent of 22° 36' is 0.4163 and the tangent of 22°42' is 0.4183. Therefore, θ is between 22° 36' and 22°42'.

Interpolation

It is frequently necessary to estimate the value of an angle to a closer approximation than is available in the table. This is equivalent to estimating between table entries, and the process is called INTERPOLATION. For example, in the foregoing problem it was determined that the angle value was between 22°36' and 22°42'. The following paragraphs describe the procedure for interpolating to find a closer approximation to the value of the angle.

The following arrangement of numbers is recommended for interpolation:

$$\begin{array}{ccc} & \text{ANGLE} & \text{TANGENT} \\ 6' & \left\{\begin{array}{c} 22^\circ 36' \\ \theta \\ 22^\circ 42' \end{array}\right. & \left.\begin{array}{l} \left.\begin{array}{c} 0.4163 \\ 0.4167 \end{array}\right\} .0004 \\ 0.4183 \end{array}\right\} .0020 \end{array}$$

The spread between 22°36' and 22°42' is 6', and we use the comparison of the tangent values to determine how much of this 6' spread is included in θ, the angle whose value is sought. Notice that the tangent of θ is different from tan 22°36' by only 0.0004, and the total spread in the tangent values is 0.0020. Therefore, the tangent of θ is $\frac{0.0004}{0.0020}$ of the way between the tangents of the two angles given in the table. This is 1/5 of the total spread, since

$$\frac{0.0004}{0.0020} = \frac{4}{20} = \frac{1}{5}$$

Another way of arriving at this result is to observe that the total spread is 20 ten-thousandths, and that the partial spread corresponding to angle θ is 4 ten-thousandths. Since 4 out of 20 is the same as 1 out of 5, angle θ is 1/5 of the way between 22°36' and 22°42'.

Taking 1/5 of the 6' spread between the angles, we have the following calculation:

$$\frac{1}{5} \times 6' = \frac{1}{5} \times 5'60''$$

$$= 1'12'' \text{ (1 minute and 12 seconds)}$$

The 12" obtained in this calculation causes our answer to appear to have greater accuracy than the tables from which it is derived. This apparent increase in accuracy is a normal result of interpolation. Final answers based on interpolated data should be rounded off to the same degree of accuracy as that of the original data.

The value of 1 minute and 12 seconds found in the foregoing problem is added to 22°36', as follows:

$$\theta = 22^\circ 36' + 1'12'' = 22^\circ 37'12''$$

Therefore θ is 22°37', approximately.

The foregoing problem could have been solved in terms of tenths and hundredths of a degree, rather than minutes, as follows:

$$\begin{array}{cll} & \text{ANGLE} & \text{TANGENT} \\ 0.1^\circ & \left\{\begin{array}{l}\left.\begin{array}{ll}22.60^\circ & 0.4163\\ \theta & 0.4167\end{array}\right\}0.0004\\[2ex] \begin{array}{ll}22.70^\circ & 0.4183\end{array}\end{array}\right\}0.0020 & \end{array}$$

In this example, we are concerned with an angular spread of 0.10° and θ is located 1/5 of the way through this spread. Thus we have

$$\theta = 22.60^\circ + \left(\frac{1}{5} \times 0.10^\circ\right)$$
$$\theta = 22.60^\circ + 0.02^\circ$$
$$\theta = 22.62^\circ$$

Interpolation must be approached with commonsense, in order to avoid applying corrections in the wrong direction. For example, the cosine of an angle decreases in value as the angle increases from 0° to 90°. If we need the value of the cosine of an angle such as 22°39', the calculation is as follows:

$$\begin{array}{cll} & \text{ANGLE} & \text{COSINE} \\ 6' & \left\{\begin{array}{l}\left.\begin{array}{l}22^\circ 36'\\ 22^\circ 39'\end{array}\right\}3'\\[2ex] 22^\circ 42'\end{array}\right. & \left.\begin{array}{l}0.9232\\[4ex] 0.9225\end{array}\right\}0.0007 \end{array}$$

In this example, it is easy to see that 22°39' is halfway between 22°36' and 22°42'. Therefore the cosine of 22°39' is halfway between the cosine of 22°36' and that of 22°42'. Taking one-half of the spread between these cosines, we then SUBTRACT from 0.9232 to find the cosine of 22°39', as follows:

$$\begin{aligned}\cos 22^\circ 39' &= 0.9232 - \left(\frac{1}{2} \times 0.0007\right)\\ &= 0.9232 - 0.00035\\ &= 0.92285\\ &= 0.9229 \text{ (approximately)}\end{aligned}$$

Practice problems:

1. Use the table in appendix II to find the decimal value of each of the following ratios:

a. tan 45° d. sin 37°14'
b. sin 60° e. cos 51.5°
c. cos 42°6' f. tan 13.75°

2. Find the angle which corresponds to each of the following decimal values in appendix II:

a. sin θ = 0.2790 c. tan θ = 0.7604
b. cos θ = 0.9018 d. sin θ = 0.8142

Answers:

1. a. 1 d. 0.6051
 b. 0.8660 e. 0.6225
 c. 0.7420 f. 0.2447

2. a. θ = 16.2° c. θ = 37°15'
 b. θ = 25°36' d. θ = 54°30'

RIGHT TRIANGLES WITH SPECIAL ANGLES AND SIDE RATIOS

Three types of right triangles are expecially significant because of their frequent occurrence. These are the 30°-60°-90° triangle, the 45°-90° triangle, and the 3-4-5 triangle.

THE 30°-60°-90° TRIANGLE

The 30°-60°-90° triangle is so named because these are the sizes of its three angles. The sides of this triangle are in the ratio of 1 to $\sqrt{3}$ to 2, as shown in figure 19-10.

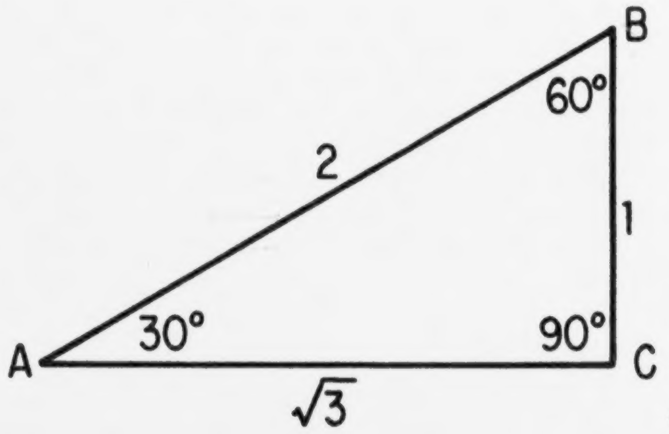


Figure 19-10.—30°-60°-90° triangle.

The sine ratio for the 30° angle in figure 19-10 establishes the proportionate values of the sides. For example, we know that the sine of 30° is 1/2; therefore side AB must be twice as long as BC. If side BC is 1 unit long, then side AB is 2 units long and, by the rule of Pythagoras, AC is found as follows:

$$AC = \sqrt{(AB)^2 - (BC)^2}$$
$$= \sqrt{4 - 1} = \sqrt{3}$$

Regardless of the size of the unit, a 30°-60°-90° triangle has a hypotenuse which is 2 times as long as the shortest side. The shortest side is opposite the 30° angle. The side opposite the 60° angle is $\sqrt{3}$ times as long as the shortest side. For example, suppose that the hypotenuse of a 30°-60°-90° triangle is 30 units long; then the shortest side is 15 units long, and the length of the side opposite the 60° angle is 15 $\sqrt{3}$ units.

Practice problems. Without reference to tables or to the rule of Pythagoras, find the following lengths and angles in figure 19-11:

1. Length of AC.
2. Size of angle A.
3. Size of angle B.
4. Length of RT.
5. Length of RS.
6. Size of angle T.

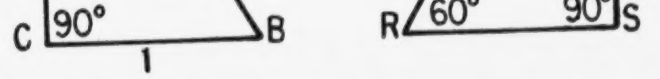


Figure 19-11.—Finding parts of 30°-60°-90° triangles.

Answers:

1. $\sqrt{3}$
2. 30°
3. 60°
4. 4
5. 2
6. 30°

THE 45°-90° TRIANGLE

Figure 19-12 illustrates a triangle in which two angles measure 45° and the third angle measures 90°. Since angles A and B are equal, the sides opposite them are also equal. Therefore, AC equals CB. Suppose that CB is 1 unit long; then AC is also 1 unit long, and the length of AB is calculated as follows:

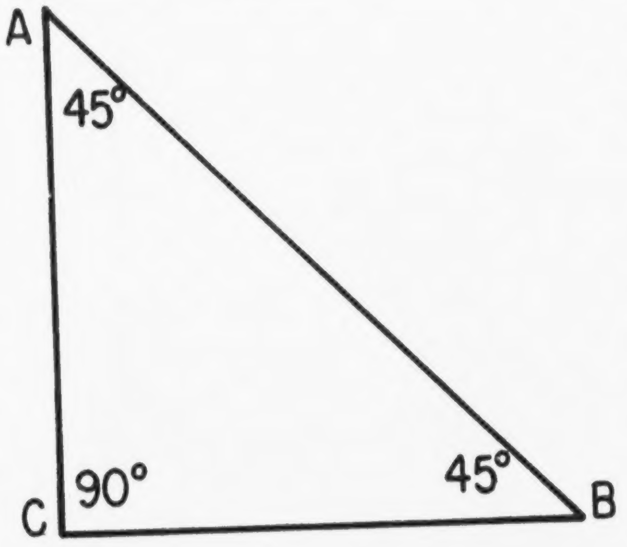


Figure 19-12.—A 45°-90° triangle.

$$(AB)^2 = 1^2 + 1^2 = 2$$
$$AB = \sqrt{2}$$

Regardless of the size of the triangle, if it has two 45° angles and one 90° angle, its sides are in the ratio 1 to 1 to $\sqrt{2}$. For example, if sides AC and CB are 3 units long, AB is 3 $\sqrt{2}$ units long.

Practice problems. Without reference to tables or to the rule of Pythagoras, find the following lengths and angles in figure 19-13:

1. AB
2. BC
3. Angle B

Answers:

1. 2 $\sqrt{2}$
2. 2
3. 45°

THE 3-4-5 TRIANGLE

The triangle shown in figure 19-14 has its sides in the ratio 3 to 4 to 5. Any triangle with its sides in this ratio is a right triangle.

It is a common error to assume that a triangle is a 3-4-5 type because two sides are known to be in the ratio 3 to 4, or perhaps 4 to 5. Figure 19-15 shows two examples of triangles which happen to have two of their sides in the stated ratio, but not the third side. This

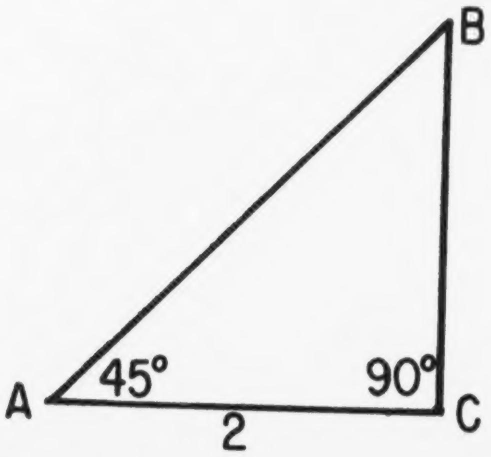


Figure 19-13.—Finding unknown parts in a 45°-90° triangle.

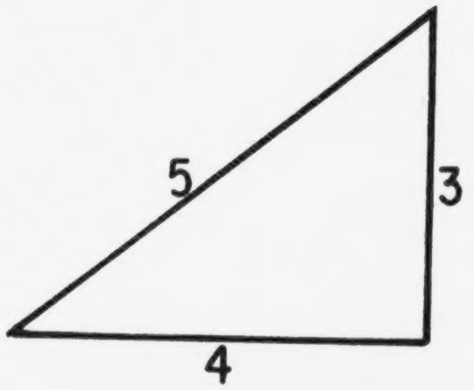
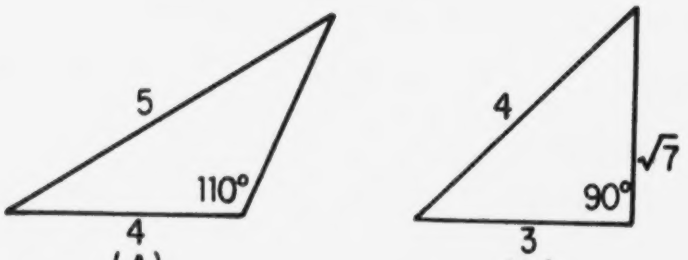


Figure 19-14.—A 3-4-5 triangle.



(A) (B)

Figure 19-15.—Triangles which may be mistaken for 3-4-5 triangles.

can be because the triangle is not a right triangle, as in figure 19-15 (A). On the other hand, even though the triangle is a right triangle its longest side may be the 4-unit side, in which case the third side cannot be 5 units long. (See fig. 19-15 (B).)

It is interesting to note that the third side in figure 19-15 (B) is $\sqrt{7}$. This is a very unusual coincidence, in which one side of a right triangle is the square root of the sum of the other two sides.

Related to the basic 3-4-5 triangle are all triangles whose sides are in the ratio 3 to 4 to 5 but are longer (proportionately) than these basic lengths. For example, the triangle pictured in figure 19-6 is a 3-4-5 triangle.

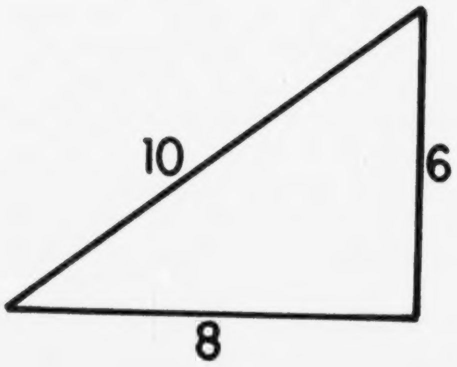


Figure 19-16.—Triangle with sides which are multiples of 3, 4, and 5.

The 3-4-5 triangle is very useful in calculations of distance. If the data can be adapted to fit a 3-4-5 configuration, no tables or calculation of square root (Pythagorean Theorem) are needed.

EXAMPLE: An observer at the top of a 40-foot vertical tower knows that the base of the tower is 30 feet from a target on the ground. How does he calculate his slant range (direct line of sight) from the target?

SOLUTION: Figure 19-17 shows that the desired length, AB, is the hypotenuse of a right triangle whose shorter sides are 30 feet and 40 feet long. Since these sides are in the ratio 3 to 4 and angle C is 90°, the triangle is a 3-4-5 triangle. Therefore, side AB represents the 5-unit side of the triangle. The ratio 30 to 40 to 50 is equivalent to 3-4-5, and thus side AB is 50 units long.

Practice problems. Without reference to tables or to the rule of Pythagoras, solve the following problems:

1. An observer is at the top of a 30-foot vertical tower. Calculate his slant range from a target on the ground which is 40 feet from the base of the tower.

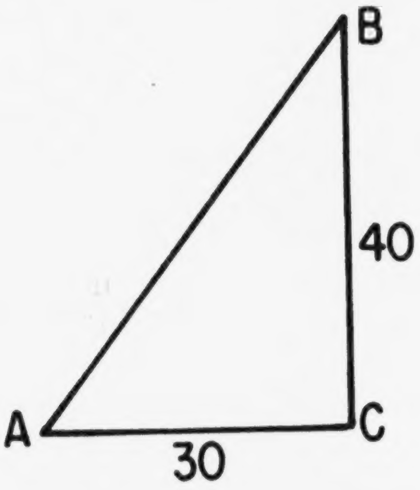


Figure 19-17.—Solving problems with a 3-4-5 triangle.

2. A guy wire 15 feet long is stretched from the top of a pole to a point on the ground 9 feet from the base of the pole. Calculate the height of the pole.

Answers:

1. 50 feet 2. 12 feet

OBLIQUE TRIANGLES

Oblique triangles were defined in chapter 17 of this training course as triangles which contain no right angles. A natural approach to the solution of problems involving oblique triangles is to construct perpendicular lines and form right triangles which subdivide the original triangle. Then the problem is solved by the usual methods for right triangles.

DIVISION INTO RIGHT TRIANGLES

The oblique triangle ABC in figure 19-18 has been divided into two right triangles by drawing line BD perpendicular to AC. The length of AC is found as follows:

1. Find the length of AD.

$$\frac{AD}{35} = \cos 40^\circ$$

$$AD = 35 \cos 40^\circ$$

$$= 35\ (0.7660)$$

$$= 26.8 \text{ (approximately)}$$

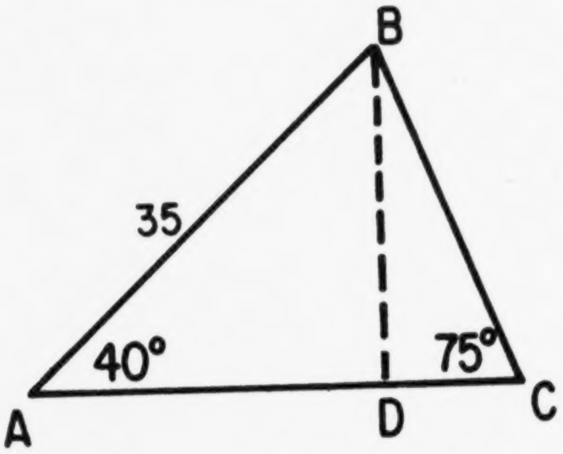


Figure 19-18.—Finding the unknown parts of an oblique triangle.

CAUTION: A careless appraisal of this problem may lead the unwary trainee to represent the ratio AC/AB as the cosine of 40°. This error is avoided only by the realization that the trigonometric ratios are based on RIGHT triangles.

2. In order to find the length of DC, first calculate BD.

$$\frac{BD}{35} = \sin 40^\circ$$

$$BD = 35 \sin 40^\circ$$

$$= 35\ (0.6428)$$

$$= 22.4 \text{ (approximately)}$$

3. Find the length of DC.

$$\frac{22.4}{DC} = \tan 75^\circ$$

$$DC = \frac{22.4}{\tan 75^\circ} = \frac{22.4}{3.732}$$

$$DC = 6.01 \text{ (approximately)}$$

4. Add AD and DC to find AC.

$$26.8 + 6.01 = 32.81$$

$$AC = 32.8 \text{ (approximately)}$$

SOLUTION BY SIMULTANEOUS EQUATIONS

A typical problem in trigonometry is the determination of the height of a point such as B in figure 19-19.

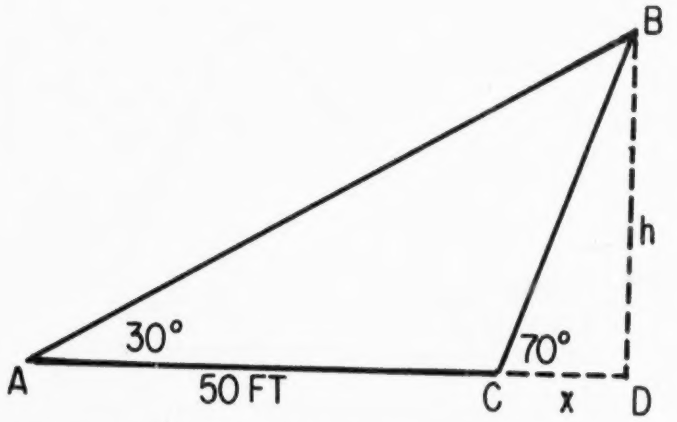


Figure 19-19.—Calculation of unknown quantities by means of oblique triangles.

Suppose that point B is the top of a hill, and point D is inaccessible. Then the only measurements possible on the ground are those shown in figure 19-19. If we let h represent BD and x represent CD, we can set up the following system of simultaneous equations:

$$\frac{h}{x} = \tan 70^\circ$$

$$\frac{h}{50 + x} = \tan 30^\circ$$

Solving these two equations for h in terms of x, we have

$$h = x \tan 70^\circ$$

and

$$h = (50 + x) \tan 30^\circ$$

Since the two quantities which are both equal to h must be equal to each other, we have

$$\begin{aligned} x \tan 70^\circ &= (50 + x) \tan 30^\circ \\ x\ (2.748) &= 50\ (0.5774) + x(0.5774) \\ x\ (2.748) - x\ (0.5774) &= 28.8 \\ x\ (2.171) &= 28.8 \\ x &= \frac{28.8}{2.171} = 13.3 \text{ feet} \end{aligned}$$

Knowing the value of x, it is now possible to compute h as follows:

$$\begin{aligned} h &= x \tan 70^\circ \\ &= 13.3\ (2.748) \\ &= 36.5 \text{ feet (approximately)} \end{aligned}$$

Practice problems:

1. Find the length of side BC in figure 19-20 (A).

2. Find the height of point B above line AD in figure 19-20 (B).

Answers:

1. 21.3 feet	2. 41.7 feet

LAW OF SINES

The law of sines provides a direct approach to the solution of oblique triangles, avoiding the necessity of subdividing into right triangles. Let the triangle in figure 19-21 (A) represent any oblique triangle with all of its angles acute.

The labels used in figure 19-21 are standardized. The small letter a is used for the side opposite angle A; small b is opposite angle B; small c is opposite angle C.

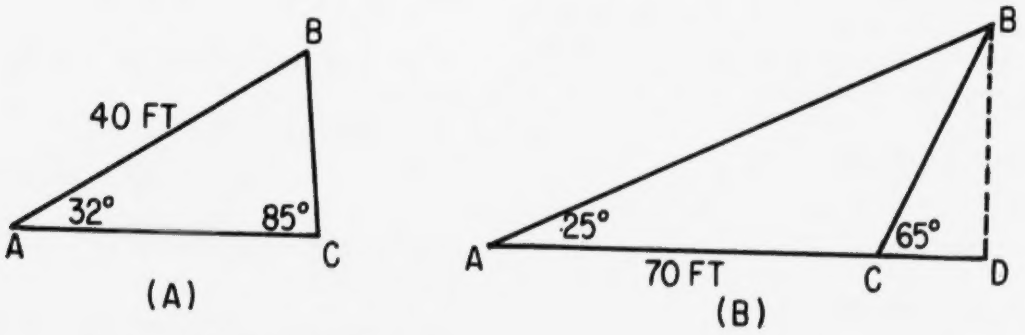


Figure 19-20.—(A) Oblique triangle with all angles acute; (B) obtuse triangle.

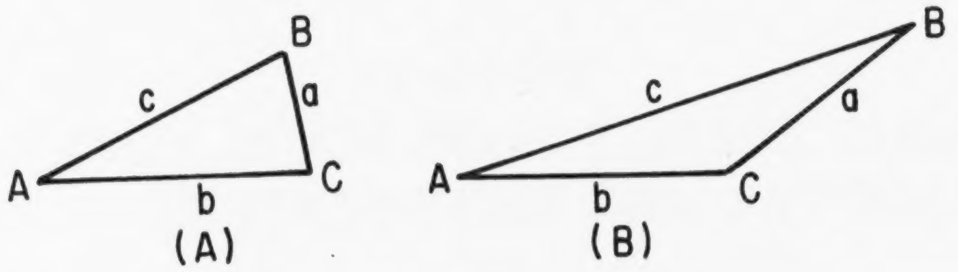


Figure 19-21.—(A) Acute oblique triangle with standard labels; (B) obtuse triangle with standard labels.

The law of sines states that in any triangle, whether it is acute as in figure 19-21 (A) or obtuse as in figure 19-21 (B), the following is true:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

EXAMPLE: In figure 19-21 (A), let angle A be 15° and let angle C be 85°. If BC is 20 units, find the length of AB.

SOLUTION: By the law of sines,

$$\frac{20}{\sin 15^\circ} = \frac{c}{\sin 85^\circ}$$

$$c = \frac{20 \sin 85^\circ}{\sin 15^\circ}$$

$$c = \frac{20\ (0.9962)}{0.2588} = 77.0$$

APPENDIX I

SQUARES, CUBES, SQUARE ROOTS, CUBE ROOTS, LOGARITHMS, AND RECIPROCALS OF NUMBERS

No.	Square	Cube	Square Root	Cube Root	Log.	1000 x Recip.	No. = Dia.	
							Circum.	Area
1	1	1	1.0000	1.0000	0.00000	1000.000	3.142	0.7854
2	4	8	1.4142	1.2599	0.30103	500.000	6.283	3.1416
3	9	27	1.7321	1.4422	0.47712	333.333	9.425	7.0686
4	16	64	2.0000	1.5874	0.60206	250.000	12.566	12.5664
5	25	125	2.2361	1.7100	0.69897	200.000	15.708	19.6350
6	36	216	2.4495	1.8171	0.77815	166.667	18.850	28.2743
7	49	343	2.6458	1.9129	0.84510	142.857	21.991	38.4845
8	64	512	2.8284	2.0000	0.90308	125.000	25.133	50.2655
9	81	729	3.0000	2.0801	0.95424	111.111	28.274	63.6173
10	100	1000	3.1623	2.1544	1.00000	100.000	31.416	78.5398
11	121	1331	3.3166	2.2240	1.04139	90.9091	34.558	95.0332
12	144	1728	3.4641	2.2894	1.07918	83.3333	37.699	113.097
13	169	2197	3.6056	2.3513	1.11394	76.9231	40.841	132.732
14	196	2744	3.7417	2.4101	1.14613	71.4286	43.982	153.938
15	225	3375	3.8730	2.4662	1.17609	66.6667	47.124	176.715
16	256	4096	4.0000	2.5198	1.20412	62.5000	50.265	201.062
17	289	4913	4.1231	2.5713	1.23045	58.8235	53.407	226.980
18	324	5832	4.2426	2.6207	1.25527	55.5556	56.549	254.469
19	361	6859	4.3589	2.6684	1.27875	52.6316	59.690	283.529
20	400	8000	4.4721	2.7144	1.30103	50.0000	62.832	314.159
21	441	9261	4.5826	2.7589	1.32222	47.6190	65.973	346.361
22	484	10648	4.6904	2.8020	1.34242	45.4545	69.115	380.133
23	529	12167	4.7958	2.8439	1.36173	43.4783	72.257	415.476
24	576	13824	4.8990	2.8845	1.38021	41.6667	75.398	452.389
25	625	15625	5.0000	2.9240	1.39794	40.0000	78.540	490.874
26	676	17576	5.0990	2.9625	1.41497	38.4615	81.681	530.929
27	729	19683	5.1962	3.0000	1.43136	37.0370	84.823	572.555
28	784	21952	5.2915	3.0366	1.44716	35.7143	87.965	615.752
29	841	24389	5.3852	3.0723	1.46240	34.4828	91.106	660.520
30	900	27000	5.4772	3.1072	1.47712	33.3333	94.248	706.858
31	961	29791	5.5678	3.1414	1.49136	32.2581	97.389	754.768
32	1024	32768	5.6569	3.1748	1.50515	31.2500	100.531	804.248
33	1089	35937	5.7446	3.2075	1.51851	30.3030	103.673	855.299
34	1156	39304	5.8310	3.2396	1.53148	29.4118	106.814	907.920
35	1225	42875	5.9161	3.2711	1.54407	28.5714	109.956	962.113
36	1296	46656	6.0000	3.3019	1.55630	27.7778	113.097	1017.88
37	1369	50653	6.0828	3.3322	1.56820	27.0270	116.239	1075.21
38	1444	54872	6.1644	3.3620	1.57978	26.3158	119.381	1134.11
39	1521	59319	6.2450	3.3912	1.59106	25.6410	122.522	1194.59
40	1600	64000	6.3246	3.4200	1.60206	25.0000	125.66	1256.64
41	1681	68921	6.4031	3.4482	1.61278	24.3902	128.81	1320.25
42	1764	74088	6.4807	3.4760	1.62325	23.8095	131.95	1385.44
43	1849	79507	6.5574	3.5034	1.63347	23.2558	135.09	1452.20
44	1936	85184	6.6332	3.5303	1.64345	22.7273	138.23	1520.53

No.	Square	Cube	Square Root	Cube Root	Log.	1000 x Recip.	No. = Dia.	
							Circum.	Area
45	2025	91125	6.7082	3.5569	1.65321	22.2222	141.37	1590.43
46	2116	97336	6.7823	3.5830	1.66276	21.7391	144.51	1661.90
47	2209	103823	6.8557	3.6088	1.67210	21.2766	147.65	1734.94
48	2304	110592	6.9282	3.6342	1.68124	20.8333	150.80	1809.56
49	2401	117649	7.0000	3.6593	1.69020	20.4082	153.94	1885.74
50	2500	125000	7.0711	3.6840	1.69897	20.0000	157.08	1963.50
51	2601	132651	7.1414	3.7084	1.70757	19.6078	160.22	2042.82
52	2704	140608	7.2111	3.7325	1.71600	19.2308	163.36	2123.72
53	2809	148877	7.2801	3.7563	1.72428	18.8679	166.50	2206.18
54	2916	157464	7.3485	3.7798	1.73239	18.5185	169.65	2290.22
55	3025	166375	7.4162	3.8030	1.74036	18.1818	172.79	2375.83
56	3136	175616	7.4833	3.8259	1.74819	17.8571	175.93	2463.01
57	3249	185193	7.5498	3.8485	1.75587	17.5439	179.07	2551.76
58	3364	195112	7.6158	3.8709	1.76343	17.2414	182.21	2642.08
59	3481	205379	7.6811	3.8930	1.77085	16.9492	185.35	2733.97
60	3600	216000	7.7460	3.9149	1.77815	16.6667	188.50	2827.43
61	3721	226981	7.8102	3.9365	1.78533	16.3934	191.64	2922.47
62	3844	238328	7.8740	3.9579	1.79239	16.1290	194.78	3019.07
63	3969	250047	7.9373	3.9791	1.79934	15.8730	197.92	3117.25
64	4096	262144	8.0000	4.0000	1.80618	15.6250	201.06	3216.99
65	4225	274625	8.0623	4.0207	1.81291	15.3846	204.20	3318.31
66	4356	287496	8.1240	4.0412	1.81954	15.1515	207.35	3421.19
67	4489	300763	8.1854	4.0615	1.82607	14.9254	210.49	3525.65
68	4624	314432	8.2462	4.0817	1.83251	14.7059	213.63	3631.68
69	4761	328509	8.3066	4.1016	1.83885	14.4928	216.77	3739.28
70	4900	343000	8.3666	4.1213	1.84510	14.2857	219.91	3848.45
71	5041	357911	8.4261	4.1408	1.85126	14.0845	223.05	3959.19
72	5184	373248	8.4853	4.1602	1.85733	13.8889	226.19	4071.50
73	5329	389017	8.5440	4.1793	1.86332	13.6986	229.34	4185.39
74	5476	405224	8.6023	4.1983	1.86923	13.5135	232.48	4300.84
75	5625	421875	8.6603	4.2172	1.87506	13.3333	235.62	4417.86
76	5776	438976	8.7178	4.2358	1.88081	13.1579	238.76	4536.46
77	5929	456533	8.7750	4.2543	1.88649	12.9870	241.90	4656.63
78	6084	474552	8.8318	4.2727	1.89209	12.8205	245.04	4778.36
79	6241	493039	8.8882	4.2908	1.89763	12.6582	248.19	4901.67
80	6400	512000	8.9443	4.3089	1.90309	12.5000	251.33	5026.55
81	6561	531441	9.0000	4.3267	1.90849	12.3457	254.47	5153.00
82	6724	551368	9.0554	4.3445	1.91381	12.1951	257.61	5281.02
83	6889	571787	9.1104	4.3621	1.91908	12.0482	260.75	5410.61
84	7056	592704	9.1652	4.3795	1.92428	11.9048	263.89	5541.77
85	7225	614125	9.2195	4.3968	1.92942	11.7647	267.04	5674.50
86	7396	636056	9.2736	4.4140	1.93450	11.6279	270.18	5808.80
87	7569	658503	9.3274	4.4310	1.93952	11.4943	273.32	5944.68
88	7744	681472	9.3808	4.4480	1.94448	11.3636	276.46	6082.12
89	7921	704969	9.4340	4.4647	1.94939	11.2360	279.60	6221.14

No.	Square	Cube	Square Root	Cube Root	Log.	1000 x Recip.	No. = Dia.	
							Circum.	Area
90	8100	729000	9.4868	4.4814	1.95424	11.1111	282.74	6361.73
91	8281	753571	9.5394	4.4979	1.95904	10.9890	285.88	6503.88
92	8464	778688	9.5917	4.5144	1.96379	10.8696	289.03	6647.61
93	8649	804357	9.6437	4.5307	1.96848	10.7527	292.17	6792.91
94	8836	830584	9.6954	4.5468	1.97313	10.6383	295.31	6939.78
95	9025	857375	9.7468	4.5629	1.97772	10.5263	298.45	7088.22
96	9216	884736	9.7980	4.5789	1.98227	10.4167	301.59	7238.23
97	9409	912673	9.8489	4.5947	1.98677	10.3093	304.73	7389.81
98	9604	941192	9.8995	4.6104	1.99123	10.2041	307.88	7542.96
99	9801	970299	9.9499	4.6261	1.99564	10.1010	311.02	7697.69
100	10000	1000000	10.0000	4.6416	2.00000	10.00000	314.16	7853.98
101	10201	1030301	10.0499	4.6570	2.00432	9.90099	317.30	8011.85
102	10404	1061208	10.0995	4.6723	2.00860	9.80392	320.44	8171.28
103	10609	1092727	10.1489	4.6875	2.01284	9.70874	323.58	8332.29
104	10816	1124864	10.1980	4.7027	2.01703	9.61538	326.73	8494.87
105	11025	1157625	10.2470	4.7177	2.02119	9.52381	329.87	8659.01
106	11236	1191016	10.2956	4.7326	2.02531	9.43396	333.01	8824.73
107	11449	1225043	10.3441	4.7475	2.02938	9.34579	336.15	8992.02
108	11664	1259712	10.3923	4.7622	2.03342	9.25926	339.29	9160.88
109	11881	1295029	10.4403	4.7769	2.03743	9.17431	342.43	9331.32
110	12100	1331000	10.4881	4.7914	2.04139	9.09091	345.58	9503.32
111	12321	1367631	10.5357	4.8059	2.04532	9.00901	348.72	9676.89
112	12544	1404928	10.5830	4.8203	2.04922	8.92857	351.86	9852.03
113	12769	1442897	10.6301	4.8346	2.05308	8.84956	355.00	10028.7
114	12996	1481544	10.6771	4.8488	2.05690	8.77193	358.14	10207.0
115	13225	1520875	10.7238	4.8629	2.06070	8.69565	361.28	10386.9
116	13456	1560896	10.7703	4.8770	2.06446	8.62069	364.42	10568.3
117	13689	1601613	10.8167	4.8910	2.06819	8.54701	367.57	10751.3
118	13924	1643032	10.8628	4.9049	2.07188	8.47458	370.71	10935.9
119	14161	1685159	10.9087	4.9187	2.07555	8.40336	373.85	11122.0
120	14400	1728000	10.9545	4.9324	2.07918	8.33333	376.99	11309.7
121	14641	1771561	11.0000	4.9461	2.08279	8.26446	380.13	11499.0
122	14884	1815848	11.0454	4.9597	2.08636	8.19672	383.27	11689.9
123	15129	1860867	11.0905	4.9732	2.08991	8.13008	386.42	11882.3
124	15376	1906624	11.1355	4.9866	2.09342	8.06452	389.56	12076.3
125	15625	1953125	11.1803	5.0000	2.09691	8.00000	392.70	12271.8
126	15876	2000376	11.2250	5.0133	2.10037	7.93651	395.84	12469.0
127	16129	2048383	11.2694	5.0265	2.10380	7.87402	398.98	12667.7
128	16384	2097152	11.3137	5.0397	2.10721	7.81250	402.12	12868.0
129	16641	2146689	11.3578	5.0528	2.11059	7.75194	405.27	13069.8
130	16900	2197000	11.4018	5.0653	2.11394	7.69231	408.41	13273.2
131	17161	2248091	11.4455	5.0788	2.11727	7.63359	411.55	13478.2
132	17424	2299968	11.4891	5.0916	2.12057	7.57576	414.69	13684.8
133	17689	2352637	11.5326	5.1045	2.12385	7.51880	417.83	13892.9
134	17956	2406104	11.5758	5.1172	2.12710	7.46269	420.97	14102.6

APPENDIX II

NATURAL SINES, COSINES, AND TANGENTS OF ANGLES FROM 0° to 90°

0°–14.9°

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
0	sin	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157
	cos	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999
	tan	0.0000	0.0017	0.0035	0.0052	0.0070	0.0087	0.0105	0.0122	0.0140	0.0157
1	sin	0.0175	0.0192	0.0209	0.0227	0.0244	0.0262	0.0279	0.0297	0.0314	0.0332
	cos	0.9998	0.9998	0.9998	0.9997	0.9997	0.9997	0.9996	0.9996	0.9995	0.9995
	tan	0.0175	0.0192	0.0209	0.0227	.0244	0.0262	0.0279	0.0297	0.0314	0.0332
2	sin	0.0349	0.0366	0.0384	0.0401	0.0419	0.0436	0.0454	0.0471	0.0488	0.0506
	cos	0.9994	0.9993	0.9993	0.9992	0.9991	0.9990	0.9990	0.9989	0.9988	0.9987
	tan	0.0349	0.0367	0.0384	0.0402	0.0419	0.0437	0.0454	0.0472	0.0489	0.0507
3	sin	0.0523	0.0541	0.0558	0.0576	0.0593	0.0610	0.0628	0.0645	0.0663	0.0680
	cos	0.9986	0.9985	0.9984	0.9983	0.9982	0.9981	0.9980	0.9979	0.9978	0.9977
	tan	0.0524	0.0542	0.0559	0.0577	0.0594	0.0612	0.0629	0.0647	0.0664	0.0682
4	sin	0.0698	0.0715	0.0732	0.0750	0.0767	0.0785	0.0802	0.0819	0.0837	0.0854
	cos	0.9976	0.9974	0.9973	0.9972	0.9971	0.9969	0.9968	0.9966	0.9965	0.9963
	tan	0.0699	0.0717	0.0734	0.0752	0.0769	0.0787	0.0805	0.0822	0.0840	0.0857
5	sin	0.0872	0.0889	0.0906	0.0924	0.0941	0.0958	0.0976	0.0993	0.1011	0.1028
	cos	0.9962	0.9960	0.9959	0.9957	0.9956	0.9954	0.9952	0.9951	0.9949	0.9947
	tan	0.0875	0.0892	0.0910	0.0928	0.0945	0.0963	0.0981	0.0998	0.1016	0.1033
6	sin	0.1045	0.1063	0.1080	0.1097	0.1115	0.1132	0.1149	0.1167	0.1184	0.1201
	cos	0.9945	0.9943	0.9942	0.9940	0.9938	0.9936	0.9934	0.9932	0.9930	0.9928
	tan	0.1051	0.1069	0.1086	0.1104	0.1122	0.1139	0.1157	0.1175	0.1192	0.1210
7	sin	0.1219	0.1236	0.1253	0.1271	0.1288	0.1305	0.1323	0.1340	0.1357	0.1374
	cos	0.9925	0.9923	0.9921	0.9919	0.9917	0.9914	0.9912	0.9910	0.9907	0.9905
	tan	0.1228	0.1246	0.1263	0.1281	0.1299	0.1317	0.1334	0.1352	0.1370	0.1388
8	sin	0.1392	0.1409	0.1426	0.1444	0.1461	0.1478	0.1495	0.1513	0.1530	0.1547
	cos	0.9903	0.9900	0.9898	0.9895	0.9893	0.9890	0.9888	0.9885	0.9882	0.9880
	tan	0.1405	0.1423	0.1441	0.1459	0.1477	0.1495	0.1512	0.1530	0.1548	0.1566
9	sin	0.1564	0.1582	0.1599	0.1616	0.1633	0.1650	0.1668	0.1685	0.1702	0.1719
	cos	0.9877	0.9874	0.9871	0.9869	0.9866	0.9863	0.9860	0.9857	0.9854	0.9851
	tan	0.1584	0.1602	0.1620	0.1638	0.1655	0.1673	0.1691	0.1709	0.1727	0.1745
10	sin	0.1736	0.1754	0.1771	0.1788	0.1805	0.1822	0.1840	0.1857	0.1874	0.1891
	cos	0.9848	0.9845	0.9842	0.9839	0.9836	0.9833	0.9829	0.9826	0.9823	0.9820
	tan	0.1763	0.1781	0.1799	0.1817	0.1835	0.1853	0.1871	0.1890	0.1908	0.1926
11	sin	0.1908	0.1925	0.1942	0.1959	0.1977	0.1994	0.2011	0.2028	0.2045	0.2062
	cos	0.9816	0.9813	0.9810	0.9806	0.9803	0.9799	0.9796	0.9792	0.9789	0.9785
	tan	0.1944	0.1962	0.1980	0.1998	0.2016	0.2035	0.2053	0.2071	0.2089	0.2107
12	sin	0.2079	0.2096	0.2113	0.2130	0.2147	0.2164	0.2181	0.2198	0.2215	0.2232
	cos	0.9781	0.9778	0.9774	0.9770	0.9767	0.9763	0.9759	0.9755	0.9751	0.9748
	tan	0.2126	0.2144	0.2162	0.2180	0.2199	0.2217	0.2235	0.2254	0.2272	0.2290
13	sin	0.2250	0.2267	0.2284	0.2300	0.2318	0.2334	0.2351	0.2368	0.2385	0.2402
	cos	0.9744	0.9740	0.9736	0.9732	0.9728	0.9724	0.9720	0.9715	0.9711	0.9707
	tan	0.2309	0.2327	0.2345	0.2364	0.2382	0.2401	0.2419	0.2438	0.2456	0.2475
14	sin	0.2419	0.2436	0.2453	0.2470	0.2487	0.2504	0.2521	0.2538	0.2554	0.2571
	cos	0.9703	0.9699	0.9694	0.9690	0.9686	0.9681	0.9677	0.9673	0.9668	0.9664
	tan	0.2493	0.2512	0.2530	0.2549	0.2568	0.2586	0.2605	0.2623	0.2642	0.2661
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

15°–29.9°

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
15	sin	0.2588	0.2605	0.2622	0.2639	0.2656	0.2672	0.2689	0.2706	0.2723	0.2740
	cos	0.9569	0.9655	0.9650	0.9646	0.9641	0.9636	0.9632	0.9627	0.9622	0.9617
	tan	0.2679	0.2698	0.2717	0.2736	0.2754	0.2773	0.2792	0.2811	0.2830	0.2849
16	sin	0.2756	0.2773	0.2790	0.2807	0.2823	0.2840	0.2857	0.2874	0.2890	0.2907
	cos	0.9613	0.9608	0.9603	0.9598	0.9593	9.9588	0.9583	0.9578	0.9573	0.9568
	tan	0.2867	0.2886	0.2905	0.2924	0.2943	0.2962	0.2981	0.3000	0.3019	0.3038
17	sin	0.2924	0.2940	0.2957	0.2974	0.2990	0.3007	0.3024	0.3040	0.3057	0.3074
	cos	0.9563	0.9558	0.9553	0.9548	0.9542	0.9537	0.9532	0.9527	0.9521	0.9516
	tan	0.3057	0.3076	0.3096	0.3115	0.3134	0.3153	0.3172	0.3191	0.3211	0.3230
18	sin	0.3090	0.3107	0.3123	0.3140	0.3156	0.3173	0.3190	0.3206	0.3223	0.3239
	cos	0.9511	0.9505	0.9500	0.9494	0.9489	0.9483	0.9478	0.9472	0.9466	0.9461
	tan	0.3249	0.3269	0.3288	0.3307	0.3327	0.3346	0.3365	0.3385	0.3404	0.3424
19	sin	0.3256	0.3272	0.3289	0.3305	0.3322	0.3338	0.3355	0.3371	0.3387	0.3404
	cos	0.9455	0.9449	0.9444	0.9438	0.9432	0.9426	0.9421	0.9415	0.9409	0.9403
	tan	0.3443	0.3463	0.3482	0.3502	0.3522	0.3541	0.3561	0.3581	0.3600	0.3620
20	sin	0.3420	0.3437	0.3453	0.3469	0.3486	0.3502	0.3518	0.3535	0.3551	0.3567
	cos	0.9397	0.9391	0.9385	0.9379	0.9373	0.9367	0.9361	0.9354	0.9348	0.9342
	tan	0.3640	0.3659	0.3679	0.3699	0.3719	0.3739	0.3759	0.3779	0.3799	0.3819
21	sin	0.3584	0.3600	0.3616	0.3633	0.3649	0.3665	0.3681	0.3697	0.3714	0.3730
	cos	0.9336	0.9330	0.9323	0.9317	0.9311	0.9304	0.9298	0.9291	0.9285	0.9278
	tan	0.3839	0.3859	0.3879	0.3899	0.3919	0.3939	0.3959	0.3979	0.4000	0.4020
22	sin	0.3746	0.3762	0.3778	0.3795	0.3811	0.3827	0.3843	0.3859	0.3875	0.3891
	cos	0.9272	0.9265	0.9259	0.9252	0.9245	0.9239	0.9232	0.9225	0.9219	0.9212
	tan	0.4040	0.4061	0.4081	0.4101	0.4122	0.4142	0.4163	0.4183	0.4204	0.4224
23	sin	0.3907	0.3923	0.3939	0.3955	0.3971	0.3987	0.4003	0.4019	0.4035	0.4051
	cos	0.9205	0.9198	0.9191	0.9184	0.9178	0.9171	0.9164	0.9157	0.9150	0.9143
	tan	0.4245	0.4265	0.4286	0.4307	0.4327	0.4348	0.4369	0.4390	0.4411	0.4431
24	sin	0.4067	0.4083	0.4099	0.4115	0.4131	0.4147	0.4163	0.4179	0.4195	0.4210
	cos	0.9135	0.9128	0.9121	0.9114	0.9107	0.9100	0.9092	0.9085	0.9078	0.9070
	tan	0.4452	0.4473	0.4494	0.4515	0.4536	0.4557	0.4578	0.4599	0.4621	0.4642
25	sin	0.4226	0.4242	0.4258	0.4274	0.4289	0.4305	0.4321	0.4337	0.4352	0.4368
	cos	0.9063	0.9056	0.9048	0.9041	0.9033	0.9026	0.9018	0.9011	0.9003	0.8996
	tan	0.4663	0.4684	0.4706	0.4727	0.4748	0.4770	0.4791	0.4813	0.4834	0.4856
26	sin	0.4384	0.4399	0.4415	0.4431	0.4446	0.4462	0.4478	0.4493	0.4509	0.4524
	cos	0.8988	0.8980	0.8973	0.8965	0.8957	0.8949	0.8942	0.8934	0.8926	0.8918
	tan	0.4877	0.4899	0.4921	0.4942	0.4964	0.4986	0.5008	0.5029	0.5051	0.5073
27	sin	0.4540	0.4555	0.4571	0.4586	0.4602	0.4617	0.4633	0.4648	0.4664	0.4679
	cos	0.8910	0.8902	0.8894	0.8886	0.8878	0.8870	0.8862	0.8854	0.8846	0.8838
	tan	0.5095	0.5117	0.5139	0.5161	0.5184	0.5206	0.5228	0.5250	0.5272	0.5295
28	sin	0.4695	0.4710	0.4726	0.4741	0.4756	0.4772	0.4787	0.4802	0.4818	0.4833
	cos	0.8829	0.8821	0.8813	0.8805	0.8796	0.8788	0.8780	0.8771	0.8763	0.8755
	tan	0.5317	0.5340	0.5362	0.5384	0.5407	0.5430	0.5452	0.5475	0.5498	0.5520
29	sin	0.4848	0.4863	0.4879	0.4894	0.4909	0.4924	0.4939	0.4955	0.4970	0.4985
	cos	0.8746	0.8738	0.8729	0.8721	0.8712	0.8704	0.8695	0.8686	0.8678	0.8669
	tan	0.5543	0.5566	0.5589	0.5612	0.5635	0.5658	0.5681	0.5704	0.5727	0.5750
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

30°–44.9°

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
30	sin	0.5000	0.5015	0.5030	0.5045	0.5060	0.5075	0.5090	0.5105	0.5120	0.5135
	cos	0.8660	0.8652	0.8643	0.8634	0.8625	0.8616	0.8607	0.8599	0.8590	0.8581
	tan	0.5774	0.5797	0.5820	0.5844	0.5867	0.5890	0.5914	0.5938	0.5961	0.5985
31	sin	0.5150	0.5165	0.5180	0.5195	0.5210	0.5225	0.5240	0.5255	0.5270	0.5284
	cos	0.8572	0.8563	0.8554	0.8545	0.8536	0.8526	0.8517	0.8508	0.8499	0.8490
	tan	0.6009	0.6032	0.6056	0.6080	0.6104	0.6128	0.6152	0.6176	0.6200	0.6224
32	sin	0.5299	0.5314	0.5329	0.5344	0.5358	0.5373	0.5388	0.5402	0.5417	0.5432
	cos	0.8480	0.8471	0.8462	0.8453	0.8443	0.8434	0.8425	0.8415	0.8406	0.8396
	tan	0.6249	0.6273	0.6297	0.6322	0.6346	0.6371	0.6395	0.6420	0.6445	0.6469
33	sin	0.5446	0.5461	0.5476	0.5490	0.5505	0.5519	0.5534	0.5548	0.5563	0.5577
	cos	0.8387	0.8377	0.8368	0.8358	0.8348	0.8339	0.8329	0.8320	0.8310	0.8300
	tan	0.6494	0.6519	0.6544	0.6569	0.6594	0.6619	0.6644	0.6669	0.6694	0.6720
34	sin	0.5592	0.5606	0.5621	0.5635	0.5650	0.5664	0.5678	0.5693	0.5707	0.5721
	cos	0.8290	0.8281	0.8271	0.8261	0.8251	0.8241	0.8231	0.8221	0.8211	0.8202
	tan	0.6745	0.6771	0.6796	0.6822	0.6847	0.6873	0.6899	0.6924	0.6950	0.6976
35	sin	0.5736	0.5750	0.5764	0.5779	0.5793	0.5807	0.5821	0.5835	0.5850	0.5864
	cos	0.8192	0.8181	0.8171	0.8161	0.8151	0.8141	0.8131	0.8121	0.8111	0.8100
	tan	0.7002	0.7028	0.7054	0.7080	0.7107	0.7133	0.7159	0.7186	0.7212	0.7239
36	sin	0.5878	0.5892	0.5906	0.5920	0.5934	0.5948	0.5962	0.5976	0.5990	0.6004
	cos	0.8090	0.8080	0.8070	0.8059	0.8049	0.9039	0.8028	0.8018	0.8007	0.7997
	tan	0.7265	0.7292	0.7319	0.7346	0.7373	0.7400	0.7427	0.7454	0.7481	0.7508
37	sin	0.6018	0.6032	0.6046	0.6060	0.6074	0.6088	0.6101	0.6115	0.6129	0.6143
	cos	0.7986	0.7976	0.7965	0.7955	0.7944	0.7934	0.7923	0.7912	0.7902	0.7891
	tan	0.7536	0.7563	0.7590	0.7618	0.7646	0.7673	0.7701	0.7729	0.7757	0.7785
38	sin	0.6157	0.6170	0.6184	0.6198	0.6211	0.6225	0.6239	0.6252	0.6266	0.6280
	cos	0.7880	0.7869	0.7859	0.7848	0.7837	0.7826	0.7815	0.7804	0.7793	0.7782
	tan	0.7813	0.7841	0.7869	0.7898	0.7926	0.7954	0.7983	0.8012	0.8040	0.8069
39	sin	0.6293	0.6307	0.6320	0.6334	0.6347	0.6361	0.6374	0.6388	0.6401	0.6414
	cos	0.7771	0.7760	0.7749	0.7738	0.7727	0.7716	0.7705	0.7694	0.7683	0.7672
	tan	0.8098	0.8127	0.8156	0.8185	0.8214	0.8243	0.8273	0.8302	0.8332	0.8361
40	sin	0.6428	0.6441	0.6455	0.6468	0.6481	0.6494	0.6508	0.6521	0.6534	0.6547
	cos	0.7660	0.7649	0.7638	0.7627	0.7615	0.7604	0.7593	0.7581	0.7570	0.7559
	tan	0.8391	0.8421	0.8451	0.8481	0.8511	0.8541	0.8571	0.8601	0.8632	0.8662
41	sin	0.6561	0.6574	0.6587	0.6600	0.6613	0.6626	0.6639	0.6652	0.6665	0.6678
	cos	0.7547	0.7536	0.7524	0.7513	0.7501	0.7490	0.7478	0.7466	0.7455	0.7443
	tan	0.8693	0.8724	0.8754	0.8785	0.8816	0.8847	0.8878	0.8910	0.8941	0.8972
42	sin	0.6691	0.6704	0.6717	0.6730	0.6743	0.6756	0.6769	0.6782	0.6794	0.6807
	cos	0.7431	0.7420	0.7408	0.7396	0.7385	0.7373	0.7361	0.7349	0.7337	0.7325
	tan	0.9004	0.9036	0.9067	0.9099	0.9131	0.9163	0.9195	0.9228	0.9260	0.9293
43	sin	0.6820	0.6833	0.6845	0.6858	0.6871	0.6884	0.6896	0.6909	0.6921	0.6934
	cos	0.7314	0.7302	0.7290	0.7278	0.7266	0.7254	0.7242	0.7230	0.7218	0.7206
	tan	0.9325	0.9358	0.9391	0.9424	0.9457	0.9490	0.9523	0.9556	0.9590	0.9623
44	sin	0.6947	0.6959	0.6972	0.6984	0.6997	0.7009	0.7022	0.7034	0.7046	0.7059
	cos	0.7193	0.7181	0.7169	0.7157	0.7145	0.7133	0.7120	0.7108	0.7096	0.7083
	tan	0.9657	0.9691	0.9725	0.9759	0.9793	0.9827	0.9861	0.9896	0.9930	0.9965
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

45°–59.9°

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
45	sin	0.7071	0.7083	0.7096	0.7108	0.7120	0.7133	0.7145	0.7157	0.7169	0.7181
	cos	0.7071	0.7059	0.7046	0.7034	0.7022	0.7009	0.6997	0.6984	0.6972	0.6959
	tan	1.0000	1.0035	1.0070	1.0105	1.0141	1.0176	1.0212	1.0247	1.0283	1.0319
46	sin	0.7193	0.7206	0.7218	0.7230	0.7242	0.7254	0.7266	0.7278	0.7290	0.7302
	cos	0.6947	0.6934	0.6921	0.6909	0.6896	0.6884	0.6871	0.6858	0.6845	0.6833
	tan	1.0355	1.0392	1.0428	1.0464	1.0501	1.0538	1.0575	1.0612	1.0649	1.0686
47	sin	0.7314	0.7325	0.7337	0.7349	0.7361	0.7373	0.7385	0.7396	0.7408	0.7420
	cos	0.6820	0.6807	0.6794	0.6782	0.6769	0.6756	0.6743	0.6730	0.6717	0.6704
	tan	1.0724	1.0761	1.0799	1.0837	1.0875	1.0913	1.0951	1.0990	1.1028	1.1067
48	sin	0.7431	0.7443	0.7455	0.7466	0.7478	0.7490	0.7501	0.7513	0.7524	0.7536
	cos	0.6691	0.6678	0.6665	0.6652	0.6639	0.6626	0.6613	0.6600	0.6587	0.6574
	tan	1.1106	1.1145	1.1184	1.1224	1.1263	1.1303	1.1343	1.1383	1.1423	1.1463
49	sin	0.7547	0.7559	0.7570	0.7581	0.7593	0.7604	0.7615	0.7627	0.7638	0.7649
	cos	0.6561	0.6547	0.6534	0.6521	0.6508	0.6494	0.6481	0.6468	0.6455	0.6441
	tan	1.1504	1.1544	1.1585	1.1626	1.1667	1.1708	1.1750	1.1792	1.1833	1.1875
50	sin	0.7660	0.7672	0.7683	0.7694	0.7705	0.7716	0.7727	0.7738	0.7749	0.7760
	cos	0.6428	0.6414	0.6401	0.6388	0.6374	0.6361	0.6347	0.6334	0.6320	0.6307
	tan	1.1918	1.1960	1.2002	1.2045	1.2088	1.2131	1.2174	1.2218	1.2261	1.2305
51	sin	0.7771	0.7782	0.7793	0.7804	0.7815	0.7826	0.7837	0.7848	0.7859	0.7869
	cos	0.6293	0.6280	0.6266	0.6252	0.6239	0.6225	0.6211	0.6198	0.6184	0.6170
	tan	1.2349	1.2393	1.2437	1.2482	1.2527	1.2572	1.2617	1.2662	1.2708	1.2753
52	sin	0.7880	0.7891	0.7902	0.7912	0.7923	0.7934	0.7944	0.7955	0.7965	0.7976
	cos	0.6157	0.6143	0.6129	0.6115	0.6101	0.6088	0.6074	0.6060	0.6046	0.6032
	tan	1.2799	1.2846	1.2892	1.2938	1.2985	1.3032	1.3079	1.3127	1.3175	1.3222
53	sin	0.7986	0.7997	0.8007	0.8018	0.8028	0.8039	0.8049	0.8059	0.8070	0.8080
	cos	0.6018	0.6004	0.5990	0.5976	0.5962	0.5948	0.5934	0.5920	0.5906	0.5892
	tan	1.3270	1.3319	1.3367	1.3416	1.3465	1.3514	1.3564	1.3613	1.3663	1.3713
54	sin	0.8090	0.8100	0.8111	0.8121	0.8131	0.8141	0.8151	0.8161	0.8171	0.8181
	cos	0.5878	0.5864	0.5850	0.5835	0.5821	0.5807	0.5793	0.5779	0.5764	0.5750
	tan	1.3764	1.3814	1.3865	1.3916	1.3968	1.4019	1.4071	1.4124	1.4176	1.4229
55	sin	0.8192	0.8202	0.8211	0.8221	0.8231	0.8241	0.8251	0.8261	0.8271	0.8281
	cos	0.5736	0.5721	0.5707	0.5693	0.5678	0.5664	0.5650	0.5635	0.5621	0.5606
	tan	1.4281	1.4335	1.4388	1.4442	1.4496	1.4550	1.4605	1.4659	1.4715	1.4770
56	sin	0.8290	0.8300	0.8310	0.8320	0.8329	0.8339	0.8348	0.8358	0.8368	0.8377
	cos	0.5592	0.5577	0.5563	0.5548	0.5534	0.5519	0.5505	0.5490	0.5476	0.5461
	tan	1.4826	1.4882	1.4938	1.4994	1.5051	1.5108	1.5166	1.5224	1.5282	1.5340
57	sin	0.8387	0.8396	0.8406	0.8415	0.8425	0.8434	0.8443	0.8453	0.8462	0.8471
	cos	0.5446	0.5432	0.5417	0.5402	0.5388	0.5373	0.5358	0.5344	0.5329	0.5314
	tan	1.5399	1.5458	1.5517	1.5577	1.5637	1.5697	1.5757	1.5818	1.5880	1.5941
58	sin	0.8480	0.8490	0.8499	0.8508	0.8517	0.8526	0.8536	0.8545	0.8554	0.8563
	cos	0.5299	0.5284	0.5270	0.5255	0.5240	0.5225	0.5210	0.5195	0.5180	0.5165
	tan	1.6003	1.6066	1.6128	1.6191	1.6255	1.6319	1.6383	1.6447	1.6512	1.6577
59	sin	0.8572	0.8581	0.8590	0.8599	0.8607	0.8616	0.8625	0.8634	0.8643	0.8652
	cos	0.5150	0.5135	0.5120	0.5105	0.5090	0.5075	0.5060	0.5045	0.5030	0.5015
	tan	1.6643	1.6709	1.6775	1.6842	1.6909	1.6977	1.7045	1.7113	1.7182	1.7251
Degs.	**Function**	**0′**	**6′**	**12′**	**18′**	**24′**	**30′**	**36′**	**42′**	**48′**	**54′**

60°–74.9°

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
60	sin	0.8660	0.8669	0.8678	0.8686	0.8695	0.8704	0.8712	0.8721	0.8729	0.8738
	cos	0.5000	0.4985	0.4970	0.4955	0.4939	0.4924	0.4909	0.4894	0.4879	0.4863
	tan	1.7321	1.7391	1.7461	1.7532	1.7603	1.7675	1.7747	1.7820	1.7893	1.7966
61	sin	0.8746	0.8755	0.8763	0.8771	0.8780	0.8788	0.8796	0.8805	0.8813	0.8821
	cos	0.4848	0.4833	0.4818	0.4802	0.4787	0.4772	0.4756	0.4741	0.4726	0.4710
	tan	1.8040	1.8115	1.8190	1.8265	1.8341	1.8418	1.8495	1.8572	1.8650	1.8728
62	sin	0.8829	0.8838	0.8846	0.8854	0.8862	0.8870	0.8878	0.8886	0.8894	0.8902
	cos	0.4695	0.4679	0.4664	0.4648	0.4633	0.4617	0.4602	0.4586	0.4571	0.4555
	tan	1.8807	1.8887	1.8967	1.9047	1.9128	1.9210	1.9292	1.9375	1.9458	1.9542
63	sin	0.8910	0.8918	0.8926	0.8934	0.8942	0.8949	0.8957	0.8965	0.8973	0.8980
	cos	0.4540	0.4524	0.4509	0.4493	0.4478	0.4462	0.4446	0.4431	0.4415	0.4399
	tan	1.9626	1.9711	1.9797	1.9883	1.9970	2.0057	2.0145	2.0233	2.0323	2.0413
64	sin	0.8988	0.8996	0.9003	0.9011	0.9018	0.9026	0.9033	0.9041	0.9048	0.9056
	cos	0.4384	0.4368	0.4352	0.4337	0.4321	0.4305	0.4289	0.4274	0.4258	0.4242
	tan	2.0503	2.0594	2.0686	2.0778	2.0872	2.0965	2.1060	2.1155	2.1251	2.1348
65	sin	0.9063	0.9070	0.9078	0.9085	0.9092	0.9100	0.9107	0.9114	0.9121	0.9128
	cos	0.4226	0.4210	0.4195	0.4179	0.4163	0.4147	0.4131	0.4115	0.4099	0.4083
	tan	2.1445	2.1543	2.1642	2.1742	2.1842	2.1943	2.2045	2.2148	2.2251	2.2355
66	sin	0.9135	0.9143	0.9150	0.9157	0.9164	0.9171	0.9178	0.9184	0.9191	0.9198
	cos	0.4067	0.4051	0.4035	0.4019	0.4003	0.3987	0.3971	0.3955	0.3939	0.3923
	tan	2.2460	2.2566	2.2673	2.2781	2.2889	2.2998	2.3109	2.3220	2.3332	2.3445
67	sin	0.9205	0.9212	0.9219	0.9225	0.9232	0.9239	0.9245	0.9252	0.9259	0.9265
	cos	0.3907	0.3891	0.3875	0.3859	0.3843	0.3827	0.3811	0.3795	0.3778	0.3762
	tan	2.3559	2.3673	2.3789	2.3906	2.4023	2.4142	2.4262	2.4383	2.4504	2.4627
68	sin	0.9272	0.9278	0.9285	0.9291	0.9298	0.9304	0.9311	0.9317	0.9323	0.9330
	cos	0.3746	0.3730	0.3714	0.3697	0.3681	0.3665	0.3649	0.3633	0.3616	0.3600
	tan	2.4751	2.4876	2.5002	2.5129	2.5257	2.5386	2.5517	2.5649	2.5782	2.5916
69	sin	0.9336	0.9342	0.9348	0.9354	0.9361	0.9367	0.9373	0.9379	0.9385	0.9391
	cos	0.3584	0.3567	0.3551	0.3535	0.3518	0.3502	0.3486	0.3469	0.3453	0.3437
	tan	2.6051	2.6187	2.6325	2.6464	2.6605	2.6746	2.6889	2.7034	2.7179	2.7326
70	sin	0.9397	0.9403	0.9409	0.9415	0.9421	0.9426	0.9432	0.9438	0.9444	0.9449
	cos	0.3420	0.3404	0.3387	0.3371	0.3355	0.3338	0.3322	0.3305	0.3289	0.3272
	tan	2.7475	2.7625	2.7776	2.7929	2.8083	2.8239	2.8397	2.8556	2.8716	2.8878
71	sin	0.9455	0.9461	0.9466	0.9472	0.9478	0.9483	0.9489	0.9494	0.9500	0.9505
	cos	0.3256	0.3239	0.3223	0.3206	0.3190	0.3173	0.3156	0.3140	0.3123	0.3107
	tan	2.9042	2.9208	2.9375	2.9544	2.9714	2.9887	3.0061	3.0237	3.0415	3.0595
72	sin	0.9511	0.9516	0.9521	0.9527	0.9532	0.9537	0.9542	0.9548	0.9553	0.9558
	cos	0.3090	0.3074	0.3057	0.3040	0.3024	0.3007	0.2990	0.2974	0.2957	0.2940
	tan	3.0777	3.0961	3.1146	3.1334	3.1524	3.1716	3.1910	3.2106	3.2305	3.2506
73	sin	0.9563	0.9568	0.9573	0.9578	0.9583	0.9588	0.9593	0.9598	0.9603	0.9608
	cos	0.2924	0.2907	0.2890	0.2874	0.2857	0.2840	0.2823	0.2807	0.2790	0.2773
	tan	3.2709	3.2914	3.3122	3.3332	3.3544	3.3759	3.3977	3.4197	3.4420	3.4646
74	sin	0.9613	0.9617	0.9622	0.9627	0.9632	0.9636	0.9641	0.9646	0.9650	0.9655
	cos	0.2756	0.2740	0.2723	0.2706	0.2689	0.2672	0.2656	0.2639	0.2622	0.2605
	tan	3.4874	3.5105	3.5339	3.5576	3.5816	3.6059	3.6305	3.6554	3.6806	3.7062
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

75°–89.9°

Degs.	Function	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°
75	sin	0.9659	0.9664	0.9668	0.9673	0.9677	0.9681	0.9686	0.9690	0.9694	0.9699
	cos	0.2588	0.2571	0.2554	0.2538	0.2521	0.2504	0.2487	0.2470	0.2453	0.2436
	tan	3.7321	3.7583	3.7848	3.8118	3.8391	3.8667	3.8947	3.9232	3.9520	3.9812
76	sin	0.9703	0.9707	0.9711	0.9715	0.9720	0.9724	0.9728	0.9732	0.9736	0.9740
	cos	0.2419	0.2402	0.2385	0.2368	0.2351	0.2334	0.2317	0.2300	0.2284	0.2267
	tan	4.0108	4.0408	4.0713	4.1022	4.1335	4.1653	4.1976	4.2303	4.2635	4.2972
77	sin	0.9744	0.9748	0.9751	0.9755	0.9759	0.9763	0.9767	0.9770	0.9774	0.9778
	cos	0.2250	0.2232	0.2215	0.2198	0.2181	0.2164	0.2147	0.2130	0.2113	0.2096
	tan	4.3315	4.3662	4.4015	4.4374	4.4737	4.5107	4.5483	4.5864	4.6252	4.6646
78	sin	0.9781	0.9785	0.9789	0.9792	0.9796	0.9799	0.9803	0.9806	0.9810	0.9813
	cos	0.2079	0.2062	0.2045	0.2028	0.2011	0.1994	0.1977	0.1959	0.1942	0.1925
	tan	4.7046	4.7453	4.7867	4.8288	4.8716	4.9152	4.9594	5.0045	5.0504	5.0970
79	sin	0.9816	0.9820	0.9823	0.9826	0.9829	0.9833	0.9836	0.9839	0.9842	0.9845
	cos	0.1908	0.1891	0.1874	0.1857	0.1840	0.1822	0.1805	0.1788	0.1771	0.1754
	tan	5.1446	5.1929	5.2422	5.2924	5.3435	5.3955	5.4486	5.5026	5.5578	5.6140
80	sin	0.9848	0.9851	0.9854	0.9857	0.9860	0.9863	0.9866	0.9869	0.9871	0.9874
	cos	0.1736	0.1719	0.1702	0.1685	0.1668	0.1650	0.1633	0.1616	0.1599	0.1582
	tan	5.6713	5.7297	5.7894	5.8502	5.9124	5.9758	6.0405	6.1066	6.1742	6.2432
81	sin	0.9877	0.9880	0.9882	0.9885	0.9888	0.9890	0.9893	0.9895	0.9898	0.9900
	cos	0.1564	0.1547	0.1530	0.1513	0.1495	0.1478	0.1461	0.1444	0.1426	0.1409
	tan	6.3138	6.3859	6.4596	6.5350	6.6122	6.6912	6.7720	6.8548	6.9395	7.0264
82	sin	0.9903	0.9905	0.9907	0.9910	0.9912	0.9914	0.9917	0.9919	0.9921	0.9923
	cos	0.1392	0.1374	0.1357	0.1340	0.1323	0.1305	0.1288	0.1271	0.1253	0.1236
	tan	7.1154	7.2066	7.3002	7.3962	7.4947	7.5958	7.6996	7.8062	7.9158	8.0285
83	sin	0.9925	0.9928	0.9930	0.9932	0.9934	0.9936	0.9938	0.9940	0.9942	0.9943
	cos	0.1219	0.1201	0.1184	0.1167	0.1149	0.1132	0.1115	0.1097	0.1080	0.1063
	tan	8.1443	8.2636	8.3863	8.5126	8.6427	8.7769	8.9152	9.0579	9.2052	9.3572
84	sin	0.9945	0.9947	0.9949	0.9951	0.9952	0.9954	0.9956	0.9957	0.9959	0.9960
	cos	0.1045	0.1028	0.1011	0.0993	0.0976	0.0958	0.0941	0.0924	0.0906	0.0889
	tan	9.5144	9.6768	9.8448	10.02	10.20	10.39	10.58	10.78	10.99	11.20
85	sin	0.9962	0.9963	0.9965	0.9966	0.9968	0.9969	0.9971	0.9972	0.9973	0.9974
	cos	0.0872	0.0854	0.0837	0.0819	0.0802	0.0785	0.0767	0.0750	0.0732	0.0715
	tan	11.43	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95
86	sin	0.9976	0.9977	0.9978	0.9979	0.9980	0.9981	0.9982	0.9983	0.9984	0.9985
	cos	0.0698	0.0680	0.0663	0.0645	0.0628	0.0610	0.0593	0.0576	0.0558	0.0541
	tan	14.30	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46
87	sin	0.9986	0.9987	0.9988	0.9989	0.9990	0.9990	0.9991	0.9992	0.9993	0.9993
	cos	0.0523	0.0506	0.0488	0.0471	0.0454	0.0436	0.0419	0.0401	0.0384	0.0366
	tan	19.08	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27
88	sin	0.9994	0.9995	0.9995	0.9996	0.9996	0.9997	0.9997	0.9997	0.9998	0.9998
	cos	0.0349	0.0332	0.0314	0.0297	0.0279	0.0262	0.0244	0.0227	0.0209	0.0192
	tan	28.64	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08
89	sin	0.9998	0.9999	0.9999	0.9999	0.9999	1.000	1.000	1.000	1.000	1.000
	cos	0.0175	0.0157	0.0140	0.0122	0.0105	0.0087	0.0070	0.0052	0.0035	0.0017
	tan	57.29	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0
Degs.	Function	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′

APPENDIX III

MATHEMATICAL SYMBOLS

SYMBOL	NAME OR MEANING	
+	Addition or positive value	
-	Subtraction or negative value	
±	Positive or negative value	
·	Multiplication dot (Centered; not to be mistaken for decimal point.)	
x	Multiplication symbol	
( )	Parentheses	Grouping symbols
[ ]	Brackets	
{ }	Braces	
——	Vinculum (overscore)	
%	Percent	
÷	Division symbol	
:	Ratio symbol	
: :	Proportion symbol	
=	Equality symbol	
≠	"Not equal" symbol	
<	Less than	
≦	Less than or equal to	
>	Greater than	
≧	Greater than or equal to	
∝	"Varies directly as" or "is proportional to" (Not to be mistaken for Greek alpha (α).)	
$\sqrt{}$	Square root symbol	
$\sqrt{\qquad}$	Square root symbol with vinculum. Vinculum is made long enough to cover all factors of the number whose square root is to be taken.	
$\sqrt[n]{}$	Radical symbol. Letter n represents a number indicating which root is to be taken.	
i or j	Imaginary unit; operator j for electronics; represents $\sqrt{-1}$.	
∝	Infinity symbol	
. . .	Ellipsis. Used in series of numbers in which successive numbers are predictable by their conformance to a pattern; meaning is approximated by "etc."	
$\log_a N$	Logarithm of N to the base a.	
log N	Logarithm of N to the base 10. (understood)	
ln N	Natural or Napierian logarithm of N. Base of the natural or Napierian logarithm system.	
$\lvert X \rvert$	Absolute value of X.	
π	Pi. The ratio of the circumference of any circle to its diameter. Approximate numerical value is 22/7.	
	Therefore	
∠ or ∢	Angle	

APPENDIX IV

WEIGHTS AND MEASURES

Dry Measure

2 cups = 1 pint (pt)
2 pints = 1 quart (qt)
4 quarts = 1 gallon (gal)
8 quarts = 1 peck (pk)
4 pecks = 1 bushel (bu)

Liquid Measure

3 teaspoons (tsp) = 1 tablespoon (tbsp)
16 tablespoons = 1 cup
2 cups = 1 pint
16 fluid ounces (oz) = 1 pint
2 pints = 1 quart
4 quarts = 1 gallon
31.5 gallons = 1 barrel (bbl)
231 cubic inches = 1 gallon
7.48 gallons = 1 cubic foot (cu ft)

Weight

16 ounces = 1 pound (lb)

2,000 pounds = 1 short ton (T)

2,240 pounds = 1 long ton

Distance

12 inches = 1 foot (ft)

3 feet = 1 yard (yd)

5-1/2 yards = 1 rod (rd)

16-1/2 feet = 1 rod

1,760 yards = 1 statute mile (mi)

5,280 feet = 1 statute mile

Area

144 square inches = 1 square foot (sq ft)
9 square feet = 1 square yd (sq yd)
30-1/4 square yards = 1 square rod
160 square rods = 1 acre (A)
640 acres = 1 square mile (sq mi)

Volume

1,728 cubic inches = 1 cubic foot

27 cubic feet = 1 cubic yard (cu yd)

Counting Units

12 units = 1 dozen (doz)
12 dozens = 1 gross
144 units = 1 gross
24 sheets = 1 quire
480 sheets = 1 ream

Equivalents

1 cubic foot of water weighs 62.5 pounds (approx) = 1,000 ounces

1 gallon of water weighs 8-1/3 pounds (approx)

1 cubic foot = 7.48 gallons

1 inch = 2.54 centimeters
1 foot = 30.4801 centimeters
1 meter = 39.37 inches
1 liter = 1.05668 quarts (liquid) = 0.90808 quart (dry)
1 nautical mile = 6,080 feet (approx)

1 fathom = 6 feet

1 shot of chain = 15 fathoms

APPENDIX V
FORMULAS

Areas

$A = s^2$	The area of a square is equal to the square of a side.
$A = \frac{b}{2} h$	The area of a triangle is equal to one half the base times the height.
$A = \pi r^2$	The area of a circle is equal to the radius squared times pi.
$A = lw$	The area of a rectangle is equal to the length times the width.
$A = Ch$	The lateral area of a cylinder is equal to the circumference of the base times the height.

Areas

$A = 4\pi r^2$	The square area of a sphere is equal to 4 times pi times the radius squared.

Volumes

$V = e^3$	The volume of a cube equals the cube of an edge.
$V = Bh$	The volume of a rectangular solid or cylinder equals the area of the base times the height.
$V = \frac{4}{3} \pi r^3$	The volume of a sphere equals $\frac{4}{3}$ pi times the radius cubed.

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